Exercise 8

A Equation of a Curve in the form of $y = \mathrm{f}(x)$

1
Solution

Given $y = \sqrt{x} - \frac{1}{4}x^2$ .... (1)

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x$$

When $x = 4$,

$$\frac{dy}{dx} = \frac{1}{2}(4)^{-\frac{1}{2}} - \frac{1}{2}(4)$$
$$= -\frac{7}{4}$$

Substitute $x = 4$ into (1).

$$y = -2$$

Equation of the tangent at $(4, -2)$ is

$$y - (-2) = -\frac{7}{4}(x - 4)$$
$$y = -\frac{7}{4}x + 5$$

The equation of the tangent at the point where $x = 4$ is $y = -\frac{7}{4}x + 5$.

2
Solution

Given $y = 2x^2 + \ln x + 5$

$$\frac{dy}{dx} = 4x + \frac{1}{x}$$

At $x = 1$,

$$\frac{dy}{dx} = 4 + 1$$
$$= 5$$

Since the gradient tangent at $x = 1$ is 5, then the gradient of normal $= -\frac{1}{5}$.

Equation of normal at (1,7) is

$$y - 7 = -\frac{1}{5}(x - 1)$$
$$y - 7 = -\frac{1}{5}x + \frac{1}{5}$$
$$y = -\frac{1}{5}x + \frac{36}{5}$$

$\therefore\ a = -\frac{1}{5}$ and $b = \frac{36}{5}$

3
Solution

(a) Given $y = \ln\left(\dfrac{\sqrt{x}}{4x+1}\right)$ $\triangleleft$ recall $\ln\left(\dfrac{a}{b}\right) = \ln a - \ln b$

$$= \ln(\sqrt{x}) - \ln(4x+1)$$

$$= \frac{1}{2}\ln x - \ln(4x+1)$$

$$\frac{dy}{dx} = \frac{1}{2x} - \frac{4}{4x+1} \text{ .... (1)}$$

When $x = 1$,

$$\frac{dy}{dx} = \frac{1}{2(1)} - \frac{4}{4(1)+1}$$

$$= -\frac{3}{10}$$

The gradient of the tangent to the curve at $x = 1$ is $-\dfrac{3}{10}$.

(b) Substitute $x = 1$ into $y = \ln\left(\dfrac{\sqrt{x}}{4x+1}\right)$.

$$y = \ln\left(\frac{\sqrt{1}}{4(1)+1}\right)$$

$$y = -\ln 5$$

Equation of the tangent at $(1, -\ln 5)$ is

$$y - (-\ln 5) = -\frac{3}{10}(x-1)$$

$$y + \ln 5 = -\frac{3}{10}x + \frac{3}{10}$$

$$y = -\frac{3}{10}x + \frac{3}{10} - \ln 5 \text{ .... (2)}$$

$\therefore$ the equation of the tangent at $P(1, -\ln 5)$ is $y = -\dfrac{3}{10}x + \dfrac{3}{10} - \ln 5$.

(c) Given that the tangent cuts the x-axis at Q, i.e $y = 0$.
Substitute $y = 0$ into (2)

$$0 = -\frac{3}{10}x + \frac{3}{10} - \ln 5$$

$$\ln 5 - \frac{3}{10} = -\frac{3}{10}x$$

$$3 - 10\ln 5 = 3x$$

$$\therefore x = 1 - \frac{10}{3}\ln 5$$

$\therefore$ the coordinates of Q are $\left(1 - \dfrac{10}{3}\ln 5, 0\right)$.

(d) From (1): $\dfrac{dy}{dx} = \dfrac{1}{2x} - \dfrac{4}{4x+1}$

$$= \frac{-4x+1}{2x(4x+1)}$$

Given that $x = k$ is the equation of a normal to the curve, then $\dfrac{dy}{dx}$ is undefined.

Let $x(-4x+1) = 0$

$$x = \frac{1}{4} \quad \text{or} \quad x = 0 \ \text{(Rejected since } x \text{ is any real positive real number)}$$

$\therefore\ k = \dfrac{1}{4}$

4
Solution

(a) Given $y = ex^{-x}$ .... (1)

$$\frac{dy}{dx} = e^{-x} - xe^{-x} \text{ .... (2)}$$

Substitute $x = 1$ into (1) and (2)

$\therefore \quad y = e^{-1}$ and $\frac{dy}{dx} = 0$

Equation of tangent at $(1, e^{-1})$ is $y = e^{-1}$.

Substitute $x = -1$ into (1) and (2)

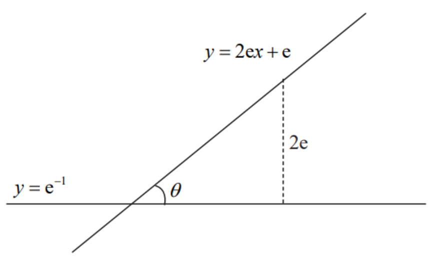
$\therefore \quad y = -e$ and $\frac{dy}{dx} = 2e$

Equation of tangent at $(1, e^{-1})$ is

$$y - (-e) = 2e(x+1)$$
$$y + e = 2ex + 2e$$
$$y = 2ex + e$$

Equation of tangent at $(1, e^{-1})$ is $y = 2ex + e$.

(b)



Let θ be the acute angle between the tangents.

$$\tan\theta = 2e$$
$$\theta = 79.6^\circ \text{ (1 dp)}$$

$\therefore$ the required angle is 79.6°.

Exercise 8

B Tangent and Normal involving Implicit Equations

5
Solution

Let $(x-1)^2-(y+4)^2=16$............................ (1)
Differentiating (1) wrt x.

$$2(x-1)-2(y+4)\frac{\mathrm{d}y}{\mathrm{d}x}=0$$
$$2(y+4)\frac{\mathrm{d}y}{\mathrm{d}x}=2(x-1)$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x-1}{y+4}$$

At $(3,\ 2)$, gradient of the tangent is $\frac{\mathrm{d}y}{\mathrm{d}x}=3$.

Thus, gradient of the normal $=\frac{-1}{\frac{\mathrm{d}y}{\mathrm{d}x}}=-3$.

Equation of the tangent at $(3,\ 2)$
$$y-2=\frac{1}{3}(x-3)$$
$$y-2=\frac{1}{3}x-1$$

Equation of the tangent at $(3,\ 2)$ is $y=\frac{1}{3}x+1$.

Equation of the normal at $(3,\ 2)$
$$y-2=-3(x-3)$$
$$y-2=-3x+9$$
$$y=-3x+11$$

Equation of the normal at $(3,\ 2)$ is $y+3x=11$.

6
Solution

(a) Given $2x - y^2 = (x+y)^2$ .... (1)

Differentiating (1) implicitly with respect to x

$$2 - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2(x+y)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right) \text{ .... (2)}$$

When the tangent is parallel to the x-axis, i.e. $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

$$2 - 2y(0) = 2(x+y)(1+0)$$
$$2 = 2(x+y)$$
$$y = 1 - x \text{ .... (3)}$$

Substituting (3) into (1).

$$2x - (1-x)^2 = (x+1-x)^2$$
$$2x - (1 - 2x + x^2) = (1)^2$$
$$x^2 - 4x + 2 = 0$$
$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)}}{2}$$
$$= 2 \pm \sqrt{2}$$

When $x = 2 - \sqrt{2}$, substituting into (2).

$$y = 1 - (2 - \sqrt{2})$$
$$= -1 + \sqrt{2}$$

When $x = 2 + \sqrt{2}$, substituting into (2).

$$y = 1 - (2 + \sqrt{2})$$
$$= -1 - \sqrt{2}$$

The equations of the tangents are $y = -1 + \sqrt{2}$ and $y = -1 - \sqrt{2}$.

(b) From (2) in **(a)**:

$$2 - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2(x+y)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$
$$2 - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2(x+y) + 2(x+y)\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$2 = 2(x+y) + 2(x+2y)\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-(x+y)}{x+2y} \text{ .... (4)}$$

Substituting $x = 2$ and $y = -2$ into (4).

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}$$

Tangent to C at $A(2,-2)$

$$y - (-2) = -\frac{1}{2}(x-2)$$
$$y + 2 = -\frac{1}{2}x + 1$$
$$y = -\frac{1}{2}x - 1 \text{ .... (5)}$$

Substituting $x = 0$ and $y = 0$ into (4).

$\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined.

$\therefore$ gradient of the normal at the origin is 0.

Hence normal to C at the origin is $y = 0$. .... (6)

When the normal and the tangent intersect, solve (5) and (6) simultaneously.

$$0 = -\frac{1}{2}x - 1$$

$$x = -2$$

$\therefore$ the coordinates of B are $(-2, 0)$.

Area of triangle OAB

$= \frac{1}{2}(2)(2)$

$= 2 \text{ units}^2$

7
Solution

(a) Let $(x+y)^2 = 4e^{xy}$ .... (1)
Differentiate (1) implicitly with respect to x.

$$2(x+y)\left(1+\frac{dy}{dx}\right) = 4e^{xy}\left(x\frac{dy}{dx}+y\right)$$

$$2(x+y)+2(x+y)\frac{dy}{dx} = 4xe^{xy}\frac{dy}{dx}+4ye^{xy}$$

$$-2xe^{xy}\frac{dy}{dx}+(x+y)\frac{dy}{dx} = 2ye^{xy}-(x+y)$$

$$\frac{dy}{dx} = \frac{2ye^{xy}-y-x}{y+x-2xe^{xy}}$$

(b) Substitute $x=0$ into (1)

$$(0+y)^2 = 4e^{(0)y}$$

$y=2$ or $y=-2$ (Rejected since the curve cuts the positive y-axis, i.e. $y>0$)

$\therefore\ A(0,-2)$

Substitute (0, 2) into $\frac{dy}{dx} = \frac{2ye^{xy}-y-x}{y+x-2xe^{xy}}$

$$\frac{dy}{dx} = \frac{2(2)-2}{2}$$
$$=1$$

Gradient of tangent at (0, 2) is 1.
Equation of the tangent at (0, 2) is

$$y-2 = 1(x-0)$$
$$y = x+2$$

Equation of the tangent to the curve at A is $y=x+2$. .... (2)

(c) Substitute (1) into (2).

$$(x+x+2)^2 = 4e^{x(x+2)}$$

$$(2x+2)^2 = 4e^{x(x+2)}$$

$$(x+1)^2 = e^{x^2+2x}$$

Using G.C.,
$x=-2$ or $x=0$ (reject since it is the x-coordinate of point A)
Substitute $x=-2$ into (2).
$\therefore\ y=0$

The coordinates of B are $(-2,\ 0)$.

8
Solution

(a) Given $\dfrac{x^2-4y^2}{x^2+xy^2}=\dfrac{1}{2}$

$$2x^2-8y^2=x^2+xy^2 \text{ .... (1)}$$

Differentiate (1) implicitly with respect to x.

$$4x-16y\frac{\mathrm{d}y}{\mathrm{d}x}=2x+\left(y^2+x2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

$$2x-y^2=\frac{\mathrm{d}y}{\mathrm{d}x}(2xy+16y)$$

$$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x-y^2}{2xy+16y} \text{ (Shown) .... (2)}$$

(b) Substitute $x=1$ into (1).

$$2(1)^2-8y^2=(1)^2+(1)y^2$$

$$2-8y^2=1+y^2$$

$$9y^2=1$$

$$y=\pm\frac{1}{3}$$

$\therefore\ P\left(1,-\dfrac{1}{3}\right)$ and $Q\left(1,\dfrac{1}{3}\right)$.

At $P\left(1,-\dfrac{1}{3}\right)$, substitute $x=1$ and $y=-\dfrac{1}{3}$ into (2).

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2-\frac{1}{9}}{-\frac{2}{3}-\frac{16}{3}}$$

$$=-\frac{17}{54}$$

Equation of tangent at $P\left(1,-\dfrac{1}{3}\right)$:

$$y+\frac{1}{3}=-\frac{17}{54}(x-1) \text{ .... (3)}$$

At $Q\left(1, \frac{1}{3}\right)$, Substitute $x = 1$ and $y = \frac{1}{3}$ into (2).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2 - \frac{1}{9}}{\frac{2}{3} + \frac{16}{3}}$$

$$= \frac{17}{54}$$

Equation of tangent at $Q\left(1, \frac{1}{3}\right)$:

$y - \frac{1}{3} = \frac{17}{54}(x-1)$ .... (4)

$(1) + (2)$:

$$2y = 0$$

$$y = 0$$

Substitute $y = 0$ into (1): $\frac{1}{3} = -\frac{17}{54}(x-1)$

$$x - 1 = -\frac{18}{17}$$

$$x = -\frac{1}{17}$$

$\therefore$ the exact coordinates of N are $\left(-\frac{1}{17}, 0\right)$.

9
Solution

(a) $x^2 - 2xy + 2y^2 = -12$

Differentiate implicitly with respect to x:

$$2x - \left(2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\right) + 4y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$2x - 2y = 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$2x - 2y = \frac{\mathrm{d}y}{\mathrm{d}x}(2x - 4y)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x - 2y}{2x - 4y}$$

$$= \frac{x - y}{x - 2y} \text{ .... (1)}$$

At $P(2, 4)$, substitute $x = 2$ and $y = 4$ into (1).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2 - 4}{2 - 8}$$

$$= \frac{1}{3}$$

Gradient of tangent at $P(2, 4)$ is $\frac{1}{3}$.

Equation of tangent at $P(2, 4)$: $y - 4 = \frac{1}{3}(x - 2)$

$$y = \frac{1}{3}x + \frac{10}{3} \text{ .... (2)}$$

Since gradient of tangent at $P(2, 4)$ is $\frac{1}{3}$. $\therefore$ gradient of normal $= -3$

Equation of normal at $P(2, 4)$: $y - 4 = -3(x - 2)$

$$y = -3x + 10 \text{ .... (3)}$$

(b) When tangent meets y-axis at A, $x = 0$.

Substitute $x = 0$ into (2): $y = \frac{10}{3}$

$\therefore A\left(0, \frac{10}{3}\right)$

When normal meets x-axis at B, $y = 0$

Substitute $y = 0$ into (3): $3x = 10$

$$x = \frac{10}{3}$$

$\therefore B\left(\frac{10}{3}, 0\right)$

$$\text{Length of } AP = \sqrt{(0-2)^2 + \left(\frac{10}{3} - 4\right)^2}$$

$$= \sqrt{\frac{40}{9}}$$

$$\text{Length of } BP = \sqrt{\left(\frac{10}{3} - 2\right)^2 + (0-4)^2}$$

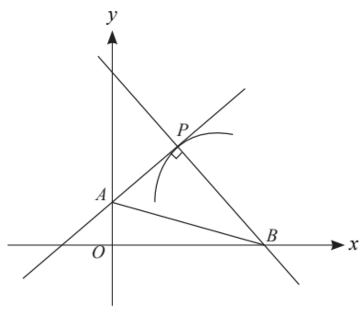
$$= \sqrt{\frac{160}{9}}$$

Area of triangle APB

$$= \frac{1}{2} \times AP \times BP$$

$$= \frac{1}{2} \times \sqrt{\frac{40}{9}} \times \sqrt{\frac{160}{9}}$$

$$= \frac{40}{9} \text{ units}^2$$



Exercise 8

B Tangent and Normal involving Parametric Equations

10
Solution

(a) Given $x = a\theta - a\sin\theta$ .... (1)
and $y = a - a\cos\theta$ .... (2)

Differentiate (1) with respect to θ

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = a - a\cos\theta$$

Differentiate (2) with respect to θ

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = a\sin\theta$$

Using the Chain Rule,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} \\ &= \frac{\mathrm{d}y}{\mathrm{d}\theta} \times \frac{\mathrm{d}\theta}{\mathrm{d}x} \\ &= a\sin\theta \times \frac{1}{a - a\cos\theta} \\ &= \frac{\sin\theta}{1-\cos\theta} \\ &= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} \quad \triangleleft \text{ use double angle formula}\end{aligned}$$

$\therefore \ \dfrac{\mathrm{d}y}{\mathrm{d}x} = \cot\dfrac{\theta}{2}$ (Shown) .... (3)

(b) Substitute $\theta = \dfrac{\pi}{3}$ into (1), (2) and (3)

From (1) $x = a\left(\dfrac{\pi}{3}\right) - a\sin\left(\dfrac{\pi}{3}\right)$

$= \left(\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{2}\right)a$

From (2) $y = a - a\cos\left(\dfrac{\pi}{3}\right)$

$= \dfrac{1}{2}a$

From (3) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \cot\dfrac{\pi}{6}$

$= \sqrt{3}$

As the gradient at $\theta = \dfrac{\pi}{3}$ is $\sqrt{3}$, $\therefore$ gradient of normal $= -\dfrac{1}{\sqrt{3}}$

Equation of normal at $\theta = \dfrac{\pi}{3}$ is

$$y - \frac{1}{2}a = -\frac{1}{\sqrt{3}}\left[x - \left(\frac{\pi}{3}a - \frac{\sqrt{3}}{2}a\right)\right]$$

$$y = -\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\left(\frac{\pi}{3}a - \frac{\sqrt{3}}{2}a\right) + \frac{1}{2}a$$

$$y = -\frac{1}{\sqrt{3}}x + \frac{a\pi}{3\sqrt{3}}$$

The equation of the normal to the curve at $\theta = \dfrac{\pi}{3}$ is $y = -\dfrac{1}{\sqrt{3}}x + \dfrac{a\pi}{3\sqrt{3}}$.

11
Solution

(a) Given $x = at^2$ .... (1)

and $y = 2at$ .... (2)

Differentiate (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2at$$

Differentiate (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2a$$

Using the Chain Rule,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}\\ &= \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}\\ &= 2a \times \frac{1}{2at}\\ &= \frac{1}{t}\end{aligned}$$

The gradient of tangent to the curve with parameter t is $\frac{1}{t}$.

$\therefore$ the gradient of normal is $-t$.

Equation of normal to the curve at any point with parameter t is

$y - 2at = -t(x - at^2)$

$\therefore$ $y + tx = 2at + at^3$ (Shown) .... (3)

(a) Substitute $t = -1$ into (3).

$y + x(-1) = 2a(-1) + a(-1)^3$

$y - x = -2a - a$

$\therefore$ $y = x - 3a$ .... (4)

Given that the normal cuts the curve again at R,

substitute (1) and (2) into (4)

$$2at = at^2 - 3a$$

$\therefore t^2 - 2t - 3 = 0$

$t = 3$ or -1 (Rejected, since it is the gradient of normal)

Substitute $t = 3$ into (1) and (2):

$x = 9a$ and $y = 6a$

$\therefore$ the coordinates of R are $(9a,\ 6a)$

(b) When the line and curve intersect,
substitute (1) and (2) into $yp = x + a$.

$$(2at)p = at^2 + a$$

$$at^2 - 2pat + a = 0 \text{ .... } (5)$$

Given that line and curve intersect cut at 2 distinct points, discriminant of (5) > 0.

i.e. $(-2pa)^2 - 4a(a) > 0$ ⊲ use $b^2 - 4ac > 0$

$$4p^2a^2 - 4a^2 > 0$$

$$(p+1)(p-1) > 0$$

$\therefore\ p > 1$ or $p < -1$

12
Solution

(a) Given $x = t + \frac{1}{t}$ .... (1)

and $y = t - \frac{1}{t}$ .... (2)

Differentiate (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - \frac{1}{t^2} = \frac{t^2 - 1}{t^2}$$

Differentiate (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 + \frac{1}{t^2} = \frac{t^2 + 1}{t^2}$$

Using the Chain Rule,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} \\ &= \frac{t^2 + 1}{t^2} \times \frac{t^2}{t^2 - 1} \\ &= \frac{t^2 + 1}{t^2 - 1}\end{aligned}$$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{t^2 + 1}{t^2 - 1}$ .... (3)

Substitute $t = p$ into (1), (2) and (3)

From (1) $\quad x = p + \frac{1}{p}$

From (2) $\quad y = p - \frac{1}{p}$

$\therefore$ the coordinates of point P are $\left(p + \frac{1}{p},\ p - \frac{1}{p}\right)$.

From (3) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{p^2 + 1}{p^2 - 1}$

The gradient at $t = p$ is $\frac{p^2 + 1}{p^2 - 1}$.

Equation of the tangent at P is

$$y-\left(p-\frac{1}{p}\right)=\frac{p^2+1}{p^2-1}\left(x-\left(p+\frac{1}{p}\right)\right) \quad \triangleleft \text{ use } y-y_1=m(x-x_1)$$

$$y-\left(\frac{p^2-1}{p}\right)=\frac{p^2+1}{p^2-1}\left(x-\frac{p^2+1}{p}\right)$$

$$p^2-1\left[y-\left(\frac{p^2-1}{p}\right)\right]=p^2+1\left(x-\frac{p^2+1}{p}\right) \quad \triangleleft \text{ multiply } (p^2-1) \text{ on both sides}$$

$$(p^2-1)y-(p^2-1)^2\left(\frac{1}{p}\right)=(p^2+1)x-(p^2+1)\left(\frac{1}{p}\right)$$

$$(p^2+1)x-(p^2-1)y=(p^2+1)^2\left(\frac{1}{p}\right)-(p^2-1)^2\left(\frac{1}{p}\right) \quad \triangleleft \text{ factor out } \frac{1}{p}$$

$$=\left(\frac{1}{p}\right)\left[(p^2+1)^2-(p^2-1)^2\right]$$

$$=\left(\frac{1}{p}\right)\left[p^4+2p^2+1-p^4+2p^2-1\right]$$

$$=\frac{1}{p}(4p^2)$$

$$=4p$$

The equation of the tangent at P is $(p^2+1)x-(p^2-1)y=4p.$ .... (4) (Shown)

(b) Let the line $y=x$ .... (5)

Given that the tangent at P meets the line $y=x$ at the point $A,$ solve (4) and (5) simultaneously to find point $A.$

Substitute (5) into (4).

$$(p^2+1)x-(p^2-1)x=4p$$

$$2x=4p$$

$$x=2p$$

Substitute $x=2p$ into (4). $\therefore\ y=2p$

Point $A(2p,2p)$

Let the line $y = x$ .... (6)

Given that the tangent at P meets the line $y = -x$ at the point B, solve (4) and (6) simultaneously to find point B.

Substitute (6) into (4).

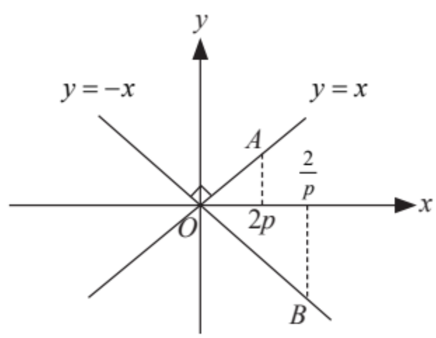
$$(p^2+1)x-(p^2-1)(-x)=4p$$

$$2p^2x=4p$$

$$x=\frac{2}{p}$$

Substitute $x=\frac{2}{p}$ into (6). $\therefore\ y=-\frac{2}{p}$

Point $B\left(\frac{2}{p},-\frac{2}{p}\right)$



Area of ΔOAB

$$=\frac{1}{2}\begin{vmatrix} 0 & 2p & \frac{2}{p} & 0 \\ 0 & 2p & -\frac{2}{p} & 0 \end{vmatrix}$$

$$=\frac{1}{2}\left|(0-4+0)-(0+4+0)\right|$$

$= 4$ units^2 (independent of p)

$\therefore$ area of triangle OAB is independent of p. (Shown)

Alternative Method

$$\text{Area}=\frac{1}{2}\left|\overline{OA}\times\overline{OB}\right|$$

$$=\frac{1}{2}\left|\begin{pmatrix} 2p \\ 2p \\ 0 \end{pmatrix}\times\begin{pmatrix} \frac{2}{p} \\ -\frac{2}{p} \\ 0 \end{pmatrix}\right|$$

$$\frac{1}{2}\left|2\cancel{p}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\times\frac{2}{\cancel{p}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right|=2\left|\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}\right|$$

$= 4$ units^2 (independent of p)

(c) $x = t + \dfrac{1}{t}$ ....(1)

$y = t - \dfrac{1}{t}$ ....(2)

$(1)+(2): x + y = 2t$ ....(3)

$(1)-(2): x - y = \dfrac{2}{t}$ ....(4)

$(3)\times(4): (x+y)(x-y) = 4$

$\therefore$ the cartesian equation is $x^2 - y^2 = 4$

From the cartesian equation : $\dfrac{x^2}{2^2} - \dfrac{y^2}{2^2} = 1$

The equation of asymptotes are $\dfrac{x^2}{2^2} - \dfrac{y^2}{2^2} = 0$

$$\frac{y^2}{2^2} = \frac{x^2}{2^2}$$

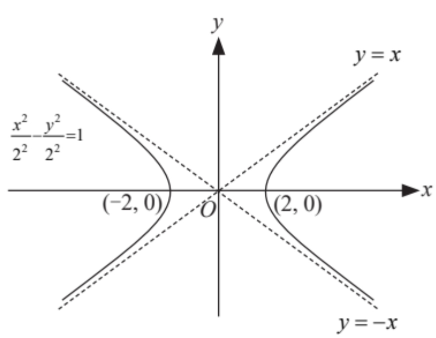
$$y = \pm x$$

To find x-intercept, let $y = 0$.

$$\frac{x^2}{2^2} = 1$$

$$x = \pm 2$$

The graph of curve C, $\dfrac{x^2}{2^2} - \dfrac{y^2}{2^2} = 1$



13
Solution

(a) Given $x = 2t + \frac{2}{t}$ .... (1)

and $y = t^2 + \frac{1}{t^2}$ .... (2)

Differentiate (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2 - \frac{2}{t^2}$$

Differentiate (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2t - \frac{2}{t^3}$$

Using the Chain Rule,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} \\ &= \frac{2t^4 - 2}{t^3} \times \frac{t^2}{2t^2 - 2} \\ &= \frac{t^2 + 1}{t} \quad \text{....................... (3)}\end{aligned}$$

Substitute x-coordinate of $P = -5$ into (1).

$$-5 = 2t + \frac{2}{t}$$

Using GC, $t = -2$.

Substitute $t = -2$ into (3): $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(-2)^2 + 1}{(-2)}$

$$= -\frac{5}{2}$$

The gradient of tangent to the curve at $P\left(-5, \frac{17}{4}\right)$ with $t = -2$ is $-\frac{5}{2}$.

$\therefore$ the gradient of tangent to the curve at $P\left(-5, \frac{17}{4}\right)$ with $t = -2$ is $\frac{2}{5}$.

Equation of normal at $P\left(-5, \frac{17}{4}\right)$ with $t = -2$ is

$$\begin{aligned}y - \frac{17}{4} &= \frac{2}{5}(x - 5) \\ y &= \frac{2}{5}x + \frac{25}{4}\end{aligned}$$

The equation of the normal to the curve at the point $P\left(-5, \frac{17}{4}\right)$ is $y = \frac{2}{5}x + \frac{25}{4}$. .... (4)

(b) When the normal meets y-axis at Y, $x = 0$.

Substitute $x = 0$ into (4): $y = \frac{25}{4}$

$\therefore Y\left(0, \frac{25}{4}\right)$

When the normal meets x-axis at X, $y = 0$

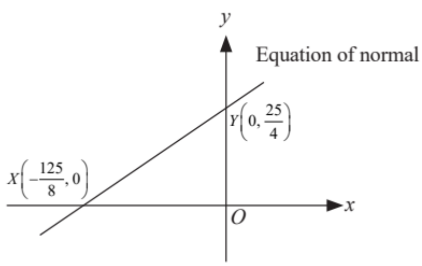
Substitute $y = 0$ into (3): $x = -\frac{125}{8}$

$\therefore X\left(-\frac{125}{8}, 0\right)$

Area of triangle OXY

$= \frac{1}{2}\left(\frac{25}{4}\right)\left(\frac{125}{8}\right)$

$= \frac{3125}{64}$ units^2



(c) Given that the normal at P cuts the curve, substitute (1) and (2) into (4).

$$t^2 + \frac{1}{t^2} = \frac{2}{5}\left(2t + \frac{2}{t}\right) + \frac{25}{4}$$

$20t^4 - 16t^3 - 125t^2 - 16t + 20 = 0$

Using G.C,

$t = 2.96$ or $-\frac{1}{2}$ (rejected $\because$ $|t| > 1$) or 0.338 (rejected $\because$ $|t| > 1$) or -2 (rejected, since it gives point P)

$\therefore\ t = 2.96$

14
Solution

(a) Given $x = \dfrac{1}{1-t^3}$

$= (1-t^3)^{-1}$..................... (1)

and $y = \dfrac{t}{1-t^3}$ .... (2)

Differentiate (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -(1-t^3)^{-2}(-3t^2)$$
$$= \frac{3t^2}{(1-t^3)^2}$$

Differentiate (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{(1-t^3)-t(-3t^2)}{(1-t^3)^2}$$
$$= \frac{1+2t^3}{(1-t^3)^2}$$

Using the Chain Rule,

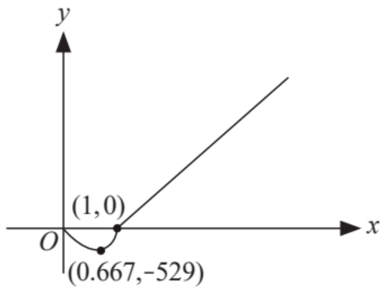
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$
$$= \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$$
$$= \frac{1+2t^3}{(1-t^3)^2} \times \frac{(1-t^3)^2}{3t^2}$$
$$= \frac{1+2t^3}{3t^2} \text{ .... (3)}$$

At stationary point of C, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

$\therefore \quad \dfrac{1+2t^3}{3t^2} = 0$

$$t = \left(-\frac{1}{2}\right)^{\frac{1}{3}}$$
$$= -0.794 \quad (3\text{ s.f.})$$

(b) The graph of curve C



(c) When $t=-1$, substitute $t=-1$ into (1), (2) and (3).

$\therefore\ x=\frac{1}{2}, y=-\frac{1}{2}$ and $\frac{dy}{dx}=-\frac{1}{3}$.

Gradient of normal $=3$.

Equation of normal at $t=-1$ is

$$y-\left(-\frac{1}{2}\right)=3\left(x-\frac{1}{2}\right)$$
$$y+\frac{1}{2}=3x-\frac{3}{2}$$
$$y=3x-2$$

$\therefore$ the equaion of the normal is $y=3x-2$. (Shown)

(d) Consider the line and the curve intersects.

Substitute (1) and (2) into (4).

$$\frac{t}{1-t^3}=3\left(\frac{1}{1-t^3}\right)-2$$
$$t=3-2(1-t^3) \quad \triangleleft \text{ multiply } (1-t^3) \text{ both sides}$$
$$2t^3-t+1=0$$
$$(t+1)(2t^2-2t+1)=0$$

$t=-1$ (Rejected, since it is the gradient of normal)

or $\quad 2t^2-2t+1=0$

Using discriminant $=(-2)^2-4(2)(1)$

$=-4<0 \Rightarrow$ no real roots

$\therefore$ there are no real roots.

There is only 1 value of t satisfying the equation, i.e. there is only 1 point of intersection.

The normal does not intersect the curve again.

15
Solution

(a) Given $x = e^{t^2-k}$ .... (1)

and $y = t^3 - kt$ .... (2)

Differentiate (1) with respect to t

$$\frac{dx}{dt} = 2te^{t^2-k}$$

Differentiate (2) with respect to t

$$\frac{dy}{dt} = 3t^2 - k$$

Using the Chain Rule,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 3t^2 - k \times \frac{1}{2te^{t^2-k}}$$

$$= \frac{3t^2 - k}{2te^{t^2-k}}$$

$$\therefore \frac{dy}{dx} = \frac{3t^2 - k}{2te^{t^2-k}}$$

(b)(i) At $t = 1$, substitute into (1), (2) and (3)

From (1) $x = e^{1-k}$

From (2) $y = 1 - k$

From (3) $\frac{dy}{dx} = \frac{3-k}{2e^{1-k}}$ ◁ gradient of tangent

Equation of tangent at $t = 1$

$$y - (1-k) = \frac{3-k}{2e^{1-k}}\left(x - e^{1-k}\right) \text{ .... (4)}$$

Given that this tangent intersects the y - axis at $y = -\frac{3}{2}$,

substitute the point $\left(0, -\frac{3}{2}\right)$ into (4).

$$-\frac{3}{2} - (1-k) = \frac{3-k}{2e^{1-k}}\left(0 - e^{1-k}\right)$$

$$-\frac{5}{2} + k = \frac{-3+k}{2}$$

$$-\frac{5}{2} + k = -\frac{3}{2} + \frac{k}{2}$$

$$k = 2 \quad \text{(Shown)}$$

(ii) When $t=1$ and $k=2$, substitute into (1), (2) and (3).

From (1) $\quad x=\mathrm{e}^{-1}$

From (2) $\quad y=-1$

From (3) $\quad \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{3(1)^2-2}{2(1)\mathrm{e}^{(1)^2-2}}$

$$=\frac{\mathrm{e}}{2}$$

Gradient of tangent is $\dfrac{\mathrm{e}}{2}$. $\therefore$ gradient of normal is $-\dfrac{2}{\mathrm{e}}$.

Equation of normal at $\left(-1,\ \dfrac{1}{\mathrm{e}}\right)$ is

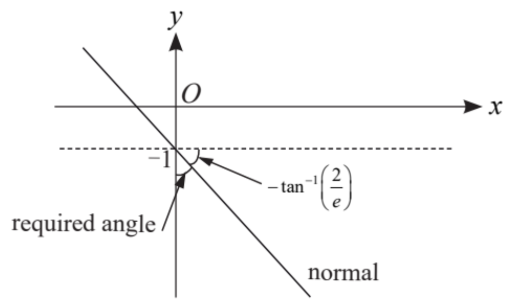
$$y-(-1)=-\frac{2}{\mathrm{e}}\left(x-\left(\frac{1}{\mathrm{e}}\right)\right)$$

$$y=-\frac{2}{\mathrm{e}}x+\left(\frac{2}{\mathrm{e}^2}-1\right)$$

(iii) Refer to the diagram.

The required angle $=\dfrac{\pi}{2}-\tan^{-1}\left(\dfrac{2}{\mathrm{e}}\right)$

$$=0.936$$



16
Solution

(a) Given $x = \dfrac{3}{t}$ .... (1)

and $y = 2t$ .... (2)

The line $y = 2x$ cuts C at the points A and B.

Substitute (1) into (2) into $y = 2x$

$$y = 2x$$
$$2t = 2\left(\frac{3}{t}\right)$$
$$t^2 = 3$$
$$t = \pm\sqrt{3}$$

When $t = \sqrt{3}$, substitute $t = \sqrt{3}$ into (1) and (2).

$\therefore\ x = \sqrt{3},\ y = 2\sqrt{3}$

The point $A(\sqrt{3},\ 2\sqrt{3})$

When $t = -\sqrt{3}$, substitute $t = -\sqrt{3}$ into (1) and (2).

$x = -\sqrt{3},\ y = -2\sqrt{3}$

The point $B(-\sqrt{3},\ -2\sqrt{3})$

Length of AB

$= \sqrt{(\sqrt{3}+\sqrt{3})^2 + (2\sqrt{3}+2\sqrt{3})^2}$ ◃ distance formula $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= 2\sqrt{15}$ units

The exact length of AB is $2\sqrt{15}$ units

(b) Differentiate (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{3}{t^2}$$

Differentiate (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2$$

Using the Chain Rule,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} \\ &= 2 \times \left(-\frac{t^2}{3}\right) \\ &= -\frac{2}{3}t^2\end{aligned}$$

$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{3}t^2$ .... (3)

At the point $P\left(\frac{3}{p}, 2p\right)$, $t = 1$.

Substitute $t = 1$ into (3).

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{2}{3}(1)^2 \\ &= -\frac{2}{3}\end{aligned}$$

Tangent at point $P\left(\frac{3}{p}, 2p\right)$ is

$$\begin{aligned}y - 2p &= -\frac{2}{3}p^2\left(x - \frac{3}{p}\right) \\ y &= -\frac{2}{3}p^2x + \frac{2p^2 \times 3}{3p} + 2p \\ y &= -\frac{2}{3}p^2x + 4p \text{ .... (4)}\end{aligned}$$

When the tangent meets the x-axis at D, i.e. $y = 0$.

Substitute $y = 0$ into (4).

At point D,

$$\begin{aligned}0 &= -\frac{2}{3}p^2x + 4p \\ \frac{2}{3}p^2x &= 4p \\ x &= \frac{6}{p}\end{aligned}$$

The point $D\left(\frac{6}{p},\ 0\right)$.

When the tangent meets the y-axis at E, i.e. $x = 0$.

Substitute $x = 0$ into (4).

At point E,

$$\begin{aligned}y &= -\frac{2}{3}p^2(0) + 4p \\ y &= 4p\end{aligned}$$

The point $E(0,\ 4p)$.

Midpoint of $DE = \left(\dfrac{\frac{6}{p}+0}{2}, \dfrac{4p+0}{2} \right)$

$$= \left(\frac{3}{p}, 2p \right)$$

$\therefore$ the coordinates of F is $\left(\dfrac{3}{p}, 2p \right)$.

From the coordinates of F,

let $x = \dfrac{3}{p}$ .... (5)

$y = 2p$

$\dfrac{y}{2} = p$ .... (6)

Substitute (6) into (5): $xy = 6$

The cartesian equation of the curve is $xy = 6$.

17
Solution

(a) Given $x = \cos^3 t$ .... (1)

and $y = \sin^3 t$ .... (2)

Differentiate (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3\cos^2 t \sin t$$

Differentiate (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3\sin^2 t \cos t$$

Using the Chain Rule,

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\
&= \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} \\
&= 3\sin^2 t \cos t \times \frac{1}{-3\cos^2 t \sin t} \\
&= \frac{3\sin^2 t \cos t}{-3\cos^2 t \sin t} \\
&= -\tan t
\end{aligned}$$

$\therefore \ \frac{\mathrm{d}y}{\mathrm{d}x} = -\tan t$

As $t \to 0$, the tangent tends towards being parallel to the x - axis. (or tangent becomes horizontal line)

As $t \to \frac{\pi}{2}$, the tangent tends towards being parallel to the y - axis. (or tangent becomes vertical line)

(b) To obtain the end-point of the curve,

Substitute $t = -\frac{\pi}{2}$ into (1) and (2). $x = 0,\ y = -1$.

Substitute $t = \pi$ into (1) and (2). $x = -1,\ y = 0$.

To find x-intercept, substitute $y = 0$ into (2). $\therefore\ t = 0,\ \pi$.

Substitute $t = 0$ into (1) and (2). $x = 1,\ y = 0$.

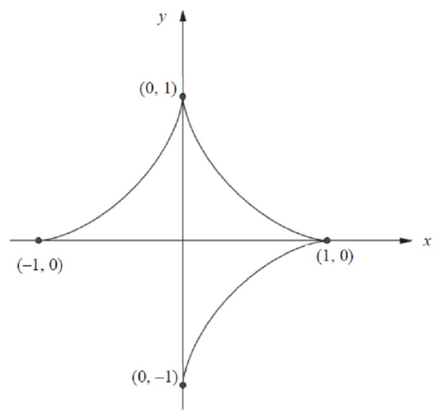
Substitute $t = \pi$ into (1) and (2). $x = -1,\ y = 0$.

To find y-intercept, substitute $x = 0$ into (1). $\therefore\ t = \frac{\pi}{2}, -\frac{\pi}{2}$.

Substitute $t = \frac{\pi}{2}$ into (1) and (2). $x = 0,\ y = 1$.

Substitute $t = -\frac{\pi}{2}$ into (1) and (2). $x = 0,\ y = -1$.

The curve for $-\frac{\pi}{2} \le t \le \pi$



(c) Equation of tangent at the point P is

$$y - \sin^3 p = -\tan p(x - \cos^3 p) \text{ ....} (3)$$

The tangent at P meets the y-axis at R, i.e. when $x = 0$.
Substitute $x = 0$ into (3)

$$y = \sin^3 p + \tan p \cos^3 p$$

$$y = \sin^3 p + \sin p \cos^2 p$$

$$y = \sin p(\sin^2 p + \cos^2 p)$$

$$= \sin p$$

Coordinates of R is $(0, \sin p)$.

The tangent at P meets the x-axis at Q, i.e. when $y = 0$.
Substitute $y = 0$ into (3)

$$0 - \sin^3 p = -\tan p(x - \cos^3 p)$$

$$x = \frac{\sin^3 p}{\tan p} + \cos^3 p$$

$$x = \cos p \sin^2 p + \cos^3 p$$

$$x = \cos p(\sin^2 p + \cos^2 p)$$

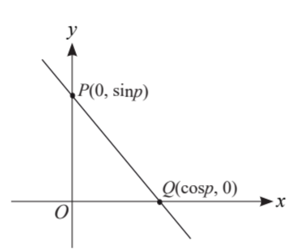
$$= \cos p$$

Coordinates of Q is $(\cos p,\ 0)$.

Area of triangle OQR

$$= \frac{1}{2}(\cos p)(\sin p) = \frac{1}{4}\sin 2p$$

$$\therefore k = \frac{1}{4}.$$



18
Solution

(a) Given $x = 2\mathrm{e}^t$ .... (1)

and $y = t^3 - t$ .... (2)

Differentiate (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2\mathrm{e}^t$$

Differentiate (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2 - 1$$

Using the Chain Rule,

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\
&= \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} \\
&= 3t^2 - 1 \times \frac{1}{2\mathrm{e}^t} \\
&= \frac{3t^2 - 1}{2\mathrm{e}^t}
\end{aligned}$$

$\therefore \ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3t^2 - 1}{2\mathrm{e}^t}$. $\therefore$ gradient of normal $= -\frac{2\mathrm{e}^t}{3t^2 - 1}$

For normals to be parallel to y-axis, i.e. $3t^2 - 1 = 0$

$$t = -\frac{1}{\sqrt{3}} \text{ or } \frac{1}{\sqrt{3}}$$

Substitute $x = -\frac{1}{\sqrt{3}}$ into (1). $\therefore x = 2\mathrm{e}^{-\frac{1}{\sqrt{3}}}$

Substitute $x = -\frac{1}{\sqrt{3}}$ into (1). $\therefore x = 2\mathrm{e}^{-\frac{1}{\sqrt{3}}}$

$\therefore$ Equations of normals parallel to the y-axis are $x = 2\mathrm{e}^{-\frac{1}{\sqrt{3}}}$ and $x = 2\mathrm{e}^{\frac{1}{\sqrt{3}}}$.

(b) To obtain the end-point of the curve,

Substitute $t = -1$ into (1) and (2). $x = 2\mathrm{e}^{-1}$, $y = 0$.

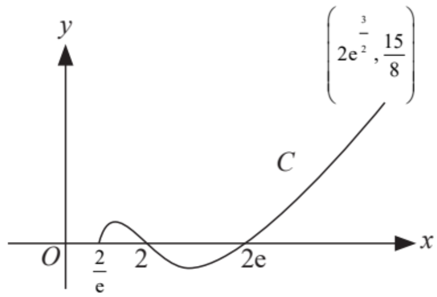
Substitute $t = \frac{3}{2}$ into (1) and (2). $x = 2\mathrm{e}^{\frac{3}{2}}$, $y = \frac{15}{8}$.

To find x-intercept, substitute $y = 0$ into (2). $\therefore \ t = -1, 1$.

Substitute $t = -1$ into (1). $x = 2\mathrm{e}^{-1}$, $y = 0$.

Substitute $t = 1$ into (1). $x = 2\mathrm{e}^2$, $y = 0$.

The curve for C, where $-1 \le t \le \frac{3}{2}$



(c) At the point $(2e^a, a^3 - a)$, when $t = a$.

Substitute $t = a$ into $\frac{dy}{dx} = \frac{3t^2 - 1}{2e^t}$.

$\therefore \frac{dy}{dx} = \frac{3a^2 - 1}{2e^a}$

Equation of the tangent at $t = a$ is

$$y - (a^3 - a) = \frac{3a^2 - 1}{2e^a}(x - 2e^a) \text{ .... } (3)$$

Given that the tangent cuts the y-axis at $y = 1$, i.e. at the coordinates (0, 1)

Substitute $x = 0$ and $y = 1$ into (3)

$$1 - (a^3 - a) = \frac{3a^2 - 1}{2e^a}(0 - 2e^a)$$

$$a(a^2 - 3a - 1) = 0$$

$\therefore a = 0$ or $a = \frac{3}{2} - \frac{\sqrt{13}}{2}$ or $a = \frac{3}{2} + \frac{\sqrt{13}}{2}$ (rejected as $-1 \le t \le \frac{3}{2}$)

Exercise 8

D Stationary Points

19
Solution

(a) Given $\mathrm{f}(x) = \mathrm{e}^x \cos x,\ 0 \le x \le \dfrac{\pi}{2}$ .... (1)

$$\mathrm{f}'(x) = \mathrm{e}^x \cos x - \mathrm{e}^x \sin x$$

At the stationary point, $\mathrm{f}'(x) = 0$.

i.e.
$$\mathrm{e}^x(\cos x - \sin x) = 0$$
$$\cos x - \sin x = 0 \quad (\text{since } \mathrm{e}^x > 0 \text{ for all } x)$$
$$\sin x = \cos x$$
$$\frac{\sin x}{\cos x} = 1$$
$$\tan x = 1$$
$$x = \frac{\pi}{4} \quad \left(\text{since } 0 \le x \le \frac{\pi}{2}\right)$$

Substitute $x = \dfrac{\pi}{4}$ into (1)

$$\mathrm{f}\left(\frac{\pi}{4}\right) = \mathrm{e}^{\frac{\pi}{4}} \cos\frac{\pi}{4}$$
$$= \frac{\sqrt{2}}{2}\mathrm{e}^{\frac{\pi}{4}}$$

$\therefore$ the stationary point of $\mathrm{f}(x)$ is $\left(\dfrac{\pi}{4}, \dfrac{\sqrt{2}}{2}\mathrm{e}^{\frac{\pi}{4}}\right)$.

Method 1 (Second Derivative Test)

$$\mathrm{f}''(x) = \mathrm{e}^x \cos x - \mathrm{e}^x \sin x - \mathrm{e}^x \sin x - \mathrm{e}^x \cos x$$
$$= -2\mathrm{e}^x \sin x$$

When $x = \dfrac{\pi}{4}$

$$\mathrm{f}''\left(\frac{\pi}{4}\right) = -2\mathrm{e}^{\frac{\pi}{4}} \sin\frac{\pi}{4}$$
$$= -\sqrt{2}\mathrm{e}^{\frac{\pi}{4}}$$

$\mathrm{e}^{\frac{\pi}{4}} > 0$ for $0 \le x \le \dfrac{\pi}{2}$

$$= -\sqrt{2}\mathrm{e}^{\frac{\pi}{4}} < 0$$

$\therefore$ the stationary point $\left(\dfrac{\pi}{4}, \dfrac{\sqrt{2}}{2}\mathrm{e}^{\frac{\pi}{4}}\right)$ is a maximum point.

Method 2 (First Derivative Test)

x	0.77	$\frac{1}{4}\pi\ (\approx 0.785)$	0.79
$f(x)$	$0.0470 > 0$	0	$-0.0143 < -$
Slope	/	—	\

As seen from the table above, stationary point $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}e^{\frac{\pi}{4}}\right)$ is a maximum point.

Learning point:
To find nature of stationary point using either second derivative test or first derivative test.

(b) Given $x = 4t^2$ .... (1)

and $y = \dfrac{2t}{t^2 + a}$ .... (2)

Differentiating (1) with respect to t

$$\frac{dx}{dt} = 8t$$

Differentiating (2) with respect to t

$$\frac{dy}{dt} = \frac{-2t^2 + 2a}{(t^2 + a)^2}$$

Using the Chain Rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{dy}{dt} \times \frac{dt}{dx} \\ &= \frac{-2t^2 + 2a}{(t^2 + a)^2} \times \frac{1}{8t} \\ &= \frac{a - t^2}{4t(t^2 + a)^2}\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{a - t^2}{4t(t^2 + a)^2}$$

At the stationary point, $\frac{dy}{dx} = 0$.

$$\frac{a - t^2}{4t(t^2 + a)^2} = 0$$

$$a - t^2 = 0$$

$$t = \sqrt{a} \quad \text{or} \quad t = -\sqrt{a}$$

Substitute $t=\sqrt{a}$ into (1) and (2)

$x=4a$ and $y=\dfrac{\sqrt{a}}{a}$

Substitute $t=-\sqrt{a}$ into (1) and (2)

$x=4a$ and $y=-\dfrac{\sqrt{a}}{a}$

$\therefore$ the stationary points are $\left(4a, \dfrac{\sqrt{a}}{a}\right)$ and $\left(4a, -\dfrac{\sqrt{a}}{a}\right)$.

Use First Derivative Test to determine the nature of the stationary points.

At stationary point $\left(4a, \dfrac{\sqrt{a}}{a}\right)$, where $t=\sqrt{a}$

t	$t=\left(\sqrt{a}\right)^-$	$t=\left(\sqrt{a}\right)$	$t=\left(\sqrt{a}\right)^+$
x	$x=(4a)^-$	$x=(4a)$	$x=(4a)^+$
sign $\dfrac{dy}{dx}$	positive	zero	negative

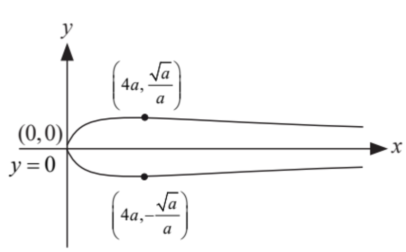
As seen from the table above, stationary point $\left(4a, \dfrac{\sqrt{a}}{a}\right)$ is a maximum.

t	$t=\left(-\sqrt{a}\right)^-$	$t=\left(-\sqrt{a}\right)$	$t=\left(-\sqrt{a}\right)^+$
x	$x=(4a)^-$	$x=(4a)$	$x=(4a)^+$
sign $\dfrac{dy}{dx}$	negative	zero	positive

As seen from the table above, stationary point $\left(4a, -\dfrac{\sqrt{a}}{a}\right)$ is a minimum.

Learning point:

The graph of the parametric equations $x=4t^2$, $y=\dfrac{2t}{t^2+a}$ is shown below.



20
Solution

Given $\dfrac{x^2-4y^2}{x^2+xy^2+100}=\dfrac{1}{2}$ .... (1)

$$2x^2-8y^2=x^2+xy^2+100$$

Differentiate implicitly with respect to x :

$$x^2-8y^2=xy^2+100$$

$$2x-16y\frac{\mathrm{d}y}{\mathrm{d}x}=2xy\frac{\mathrm{d}y}{\mathrm{d}x}+y^2$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x-y^2}{2xy+16y}$$

$\therefore \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2x-y^2}{2xy+16y}$ (shown)

Let $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0.$

$\therefore \quad \dfrac{2x-y^2}{2xy+16y}=0$

$2x=y^2$ .... (2)

Substitute (2) into (1)

$\therefore \quad \dfrac{x^2-8x}{x^2+2x^2+100}=\dfrac{1}{2}$

$$2x^2-16x=3x^2+100$$

$$x^2+16x+100=0$$

$$(x+8)^2+36=0$$

For all real values of x, $(x+8)^2\geq 0$

Since $(x+8)^2+36>0$, $\therefore$ there is no value of x.

Hence, there is no stationary point.

Alternative Method (Using discriminant)

Since $16^2-4(1)(100)=-144<0$, there is no real roots.

$\therefore$ there is no stationary point when $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ for $x\in\mathbb{R}$.

21
Solution

(a) Given $x-y=(x+y)^2$ .... (1)

Differentiate (1) implicitly with respect x:

$$1-\frac{dy}{dx}=2(x+y)\left(1+\frac{dy}{dx}\right)$$

$$1-\frac{dy}{dx}=2(x+y)+2(x+y)\frac{dy}{dx}$$

$$1-2(x+y)=2(x+y)\frac{dy}{dx}+\frac{dy}{dx}$$

$$1-2x-2y=\frac{dy}{dx}(1+2x+2y)$$

$$\frac{dy}{dx}=\frac{1-2x-2y}{1+2x+2y}$$

$$1+\frac{dy}{dx}=1+\frac{1-2x-2y}{1+2x+2y} \quad \triangleleft \text{ add 1 on both sides}$$

$$=\frac{1+2x+2y+1-2x-2y}{1+2x+2y}$$

$$1+\frac{dy}{dx}=\frac{2}{1+2x+2y} \quad \text{(Shown) .... (2)}$$

(b) From **(a)**

$$1+\frac{dy}{dx}=\frac{2}{2x+2y+1}$$

Differentiate (2) implicitly with respect x:

$$\frac{d^2y}{dx^2}=\frac{(2x+2y+1)(0)-(2)\left(2+2\frac{dy}{dx}\right)}{(2x+2y+1)^2} \quad \triangleleft \text{ use quotient rule}$$

$$=-\frac{4\left(1+\frac{dy}{dx}\right)}{(2x+2y+1)^2}$$

$$=-\frac{4}{(2x+2y+1)^2}\cdot\left(1+\frac{dy}{dx}\right)$$

$$=-\left(\frac{2}{2x+2y+1}\right)^2\cdot\left(1+\frac{dy}{dx}\right)$$

$$=-\left(1+\frac{dy}{dx}\right)^2\left(1+\frac{dy}{dx}\right)$$

$$=-\left(1+\frac{dy}{dx}\right)^3$$

$$\frac{d^2y}{dx^2}=-\left(1+\frac{dy}{dx}\right)^3 \quad \text{(Shown) .... (3)}$$

(c) At the turning point, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

Substitute $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ into (3)

$$\begin{aligned}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= -(1+0)^3 \\ &= -1 < 0\end{aligned}$$

$\therefore$ By 2nd denvative test, the turning point is a maximum.

22
Solution

(a) Given $y = 3x^4 + 6x^2 - 24x + k$ .... (1)

$$\frac{dy}{dx} = 12x^3 + 12x - 24$$

At stationary point, $\frac{dy}{dx} = 0$.

i.e. $12(x^3 + x - 2) = 0$

$\therefore \quad 12(x-1)(x^2 + x + 2) = 0$

$x = 1$ or $(x^2 + x + 2) = 0$

$$\left(x + \frac{1}{2}\right)^2 + \frac{5}{4} > 0 \text{ for real values of } x$$

$(x^2 + x + 2)$ has no solution.

Hence C has only one stationary point.

$$\frac{d^2y}{dx^2} = 36x^2 + 12$$

When $x = 2$, $\frac{d^2y}{dx^2} > 0$.

Thus the point when $x = 1$ is a minimum point.

(b) Given that the stationary point of curve C touches the x-axis, i.e. $y = 0$.

$\therefore$ the stationary point is (1, 0).

Substitute $x = 1$ and $y = 0$ into (1)

$$0 = 3(1)^4 + 6(1)^2 - 24(1) + k$$

$$0 = 3 + 6 - 24 + k$$

$$k = 15$$

$\therefore \; k = 15$

23
Solution

Given $y = ax^2 - 2bx + c$ .... (1) $\quad (a > 0)$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2ax - 2b$ .... (2)

(a) For turning point, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

i.e. $2ax - 2b = 0$

So, $x = \dfrac{b}{a}$ .... (3)

Substitute (2) into (1)

$$y = a\left(\frac{b}{a}\right)^2 - 2b\left(\frac{b}{a}\right) + c$$

$$y = c - \frac{b^2}{a} \text{ .... (4)}$$

$\therefore$ the stationary point is $\left(\dfrac{b}{a}, c - \dfrac{b^2}{a}\right)$.

Since $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2a > 0$,

$\therefore$ $\left(\dfrac{b}{a}, c - \dfrac{b^2}{a}\right)$ is a minimum turning point.

(b) Given that $\left(\dfrac{b}{a}, c - \dfrac{b^2}{a}\right)$ lies on the line $y = x$,

Substitute the point $\left(\dfrac{b}{a}, c - \dfrac{b^2}{a}\right)$ into the line $y = x$.

$\therefore$ $\dfrac{b}{a} = c - \dfrac{b^2}{a}$.

$$c = \frac{1}{a}(b^2 + b) \text{ .... (5)}$$

(c) From (5) $\quad c = \dfrac{1}{a}(b^2 + b)$ $\quad \lhd$ complete the square

$$= \frac{1}{a}\left[\left(b + \frac{1}{2}\right)^2 - \frac{1}{4}\right]$$

For all real values of b, $\left(b + \dfrac{1}{2}\right)^2$ is positive.

$$\frac{1}{a} - \frac{1}{4} \geq 0$$

$\therefore$ $c \geq -\dfrac{1}{4a}$ whatever the value of b. (Shown)

(d) Substitute the point $(0,\ 6)$ into (1).

$$6 = a(0)^2 - 2b(0) + c$$
$$c = 6$$

Given that $(2,\ 2)$ is a turning point, i.e $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 2$.

Substitute $x = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ into (2)

From (2)

$$0 = 2a(2) - 2b$$
$$\therefore \quad b = 2a \text{ .... } (6)$$

Substitute (2, 2), $c = 6$ and (6) into (4):

From (4) $\quad 2 = 6 - \frac{(2a)^2}{a}$

$$2 = 6 - 4a$$
$$a = 1$$

Substitute $a = 1$ into (6).

$\therefore\ b = 2$

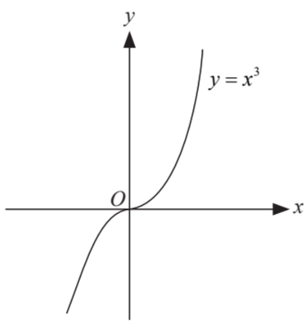
$\therefore\ a = 1,\ b = 2$ and $c = 6$.

Exercise 8

Graphical Interpretation of a Function

24
Solution

(a) The graph of $y = x^3$



From the graph, the stationary point is (0, 0).

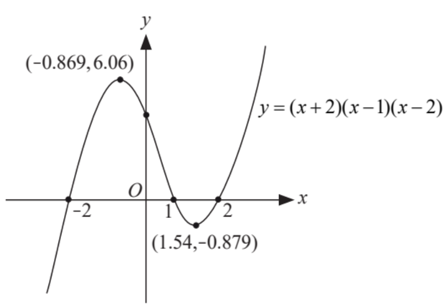
(i) The curve is strictly increasing when $x \in \mathbb{R}, x \neq 0$.

(ii) No applicable

(iii) The curve is concave upward when $x > 0$.

(iv) The curve is concave downward when $x < 0$.

(b) The graph of $y = (x+2)(x-1)(x-2)$



From the graph, the stationary points are $(1.54, -0.879)$ and $(-0.879, 6.06)$.

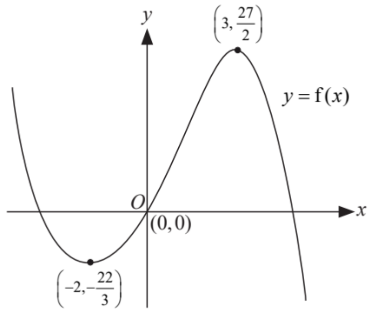
(i) The curve is strictly increasing when the range of x is $x < -0.869$ or $x > 1.54$.

(ii) The curve is strictly decreasing when the range of x is $-0.869 < x < 1.54$.

(iii) The curve is concave upward when $x > \frac{1}{3}$.

(iv) The curve is concave downward when $x < \frac{1}{3}$.

(c) The graph of $f(x) = 6x + \frac{1}{2}x^2 - \frac{1}{3}x^3$



From the graph, the stationary points are $\left(-2, -\frac{22}{3}\right)$ and $\left(3, \frac{27}{2}\right)$.

Given $f(x) = 6x + \frac{1}{2}x^2 - \frac{1}{3}x^3$

$$f'(x) = 6 + x - x^2$$

$$f''(x) = 1 - 2x$$

(i) For f is strictly decreasing, $f'(x) > 0$.

i.e. $6 + x - x^2 > 0$

$x^2 - x - 6 < 0$

$(x+2)(x-3) < 0$

$\therefore \quad -2 < x < 3$

(ii) For f is strictly decreasing, $f'(x) < 0$.

i.e. $6 + x - x^2 < 0$

$x^2 - x - 6 > 0$

$(x+2)(x-3) > 0$

$\therefore \quad x < -2 \quad$ or $\quad x > 3$

(iii) For f is concave upward, $f''(x) > 0$.

i.e. $1 - 2x > 0$

$2x - 1 < 0$

$x < \frac{1}{2}$

$\therefore \quad x < \frac{1}{2}$

(iv) For f is concave downward, $f''(x) < 0$.

i.e. $1 - 2x < 0$

$2x - 1 > 0$

$x > \frac{1}{2}$

$\therefore \quad x > \frac{1}{2}$

25
Solution

(a) $f'(x) = x^2 - 2x + 3$

$= (x-1)^2 + 2$

For all real values of x, $f'(x) > 0$.

$\therefore$ there is no stationary point.

For $f(x)$ is increasing, the range of values of x is $x \in \mathbb{R}$.

For $f(x)$ is decreasing, there is no solution of x.

(b) $f'(x) = (x+1)(x-2)(x-3)$

For stationary point of $f(x)$, i.e. $f'(x) = 0$

$\therefore \quad (x+1)(x-2)(x-3) = 0$

$x = -1, \ 2, \ 3$

The x-coordinates of the stationary points are 1, 2 and 3.

For $f(x)$ is increasing, $f'(x) \geq 0$

i.e. $(x+1)(x-2)(x-3) \geq 0$

$x \geq 3 \quad \text{or} \quad -1 \leq x \leq 2$

$\therefore \ x \geq 3 \ \text{ or } \ -1 \leq x \leq 2$

For the function $f(x)$ is decreasing, $f'(x) \leq 0$.

i.e. $\quad (x+1)(x-2)(x-3) \leq 0$

$x \leq -1 \quad \text{or} \quad 2 \leq x \leq 3$

$\therefore \ x \leq -1 \ \text{ or } \ 2 \leq x \leq 3$

26
Solution

(a) Given $y+\ln y=\mathrm{e}^x$ .... (1)

Differentiate (1) with respect to x

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\frac{\mathrm{d}y}{\mathrm{d}x}}{y}=\mathrm{e}^x \text{ .... (2)}$$

Differentiate (2) with respect to x

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}y-\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}{y^2}=\mathrm{e}^x$$

$$y^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}y-\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=y^2\mathrm{e}^x$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(y^2+y)=y^2\mathrm{e}^x+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \text{ .... (3) (Shown)}$$

(b) From (3): $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}(y^2+y)=y^2\mathrm{e}^x+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{y^2\mathrm{e}^x+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}{y^2+y}$$

From (1): $y+\ln y=\mathrm{e}^x$

Since $\ln y$ is defined, so $y>0$. Hence $y^2>0$.

$\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2>0$ since the order of derivative is even.

For all $x\in\mathbb{R}$, $\mathrm{e}^x>0$

$$\therefore\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{y^2\mathrm{e}^x+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}{y^2+y}>0$$

The curve is concave upwards for $x\in\mathbb{R}$.

27
Solution

(a) Given that $f(x)=\dfrac{x+a}{x^2-1}$ .... (1)

Differentiate (1) with respect to x

$$f'(x)=\frac{(x^2-1)-(x+a)(2x)}{(x^2-1)^2} \quad \triangleleft \text{ use quotient rule}$$

$$=\frac{-1-2ax-x^2}{(x^2-1)^2}$$

$$\therefore\ f'(x)=\frac{-1-2ax-x^2}{(x^2-1)^2}$$

(b) For f is strictly decreasing, $f'(x)<0$.

i.e. $\dfrac{-1-2ax-x^2}{(x^2-1)^2}<0$

For $x\in\mathbb{R},\ x\neq\pm1,\ (x^2-1)^2>0$.

$$\therefore \quad -1-2ax-x^2<0$$

$$x^2+2ax+1>0$$

$$(x+a)^2+1-a^2>0 \quad \triangleleft \text{ completing the square}$$

For all real values of x, $(x+a)^2$ is always positive.

Thus $\quad 1-a^2>0$

$$a^2-1<0$$

$$(a-1)(a+1)<0$$

$\therefore -1<a<1$ (Shown)

Alternative Method (Using discriminant)

$$(2a)^2-4(1)(1)<0$$

$$4a^2-4<0$$

$$a^2-1<0$$

$$(a-1)(a+1)<0$$

$\therefore -1<a<1$

28
Solution

$$y = \frac{3x^2 + 12x + \lambda}{2(x+2)} \qquad \triangleleft \text{ use long division}$$

$$= \frac{3}{2}(x+2) + \frac{\lambda - 12}{2(x+2)}$$

Differentiate y with respect to x

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2} - \frac{\lambda - 12}{2(x+2)^2}$$

$$= \frac{3(x+2)^2 - (\lambda - 12)}{2(x+2)^2}$$

For the graph of C is strictly increasing, $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.

i.e. $\frac{3(x+2)^2 - (\lambda - 12)}{2(x+2)^2} > 0$

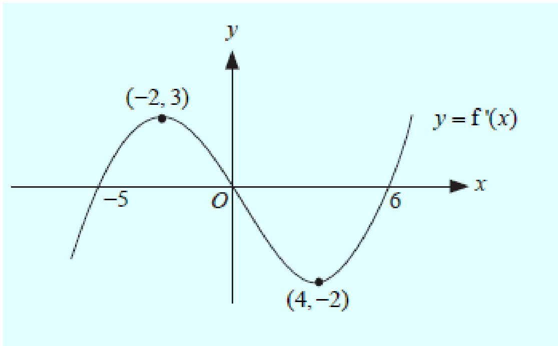
$(x+2)^2$ is always positive for all real values of x, $x \neq 2$

Thus $-(\lambda - 12) > 0$

$\lambda < 12$

The range of values of λ is $\lambda < 12$.

29
Solution



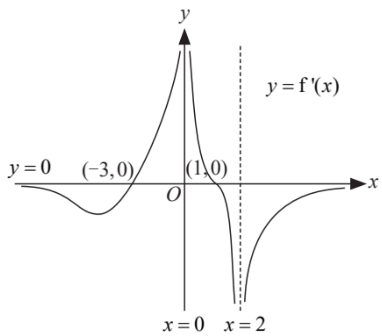
(a) The x-coordinates of the stationary points are $x = -5, 0, 6$.

(b) For $y = f(x)$ is decreasing, $f'(x) < 0$.
Refer to the diagram. It occurs when the graph $y = f'(x)$ lies below the x-axis.
$\therefore\ x < -5$ or $0 < x < 6$

(c) For the graph $y = f(x)$ is concave upwards, $f'(x)$ is increasing, i.e. $f''(x) > 0$.
Refer to the diagram. It occurs when the gradient of the graph $y = f'(x)$ is positive.
$\therefore\ x < -2$ or $x > 4$

30
Solution

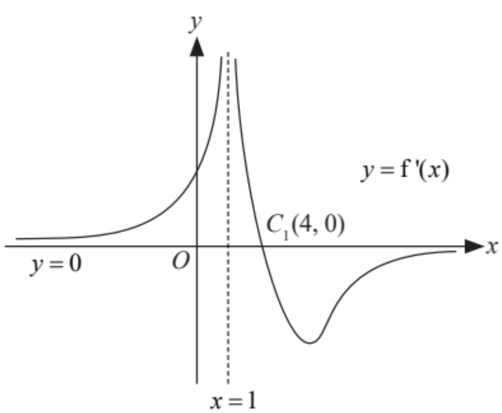
The graph of $y = f'(x)$



Learning point:
All (3) asymptotes must be labelled, and intersections with axes written in coordinate form as instructed by the question.

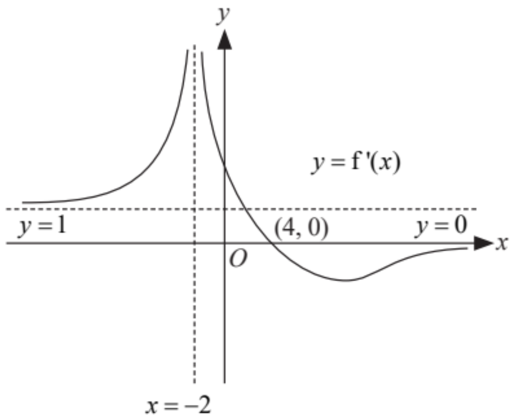
31
Solution

The graph of $y = f'(x)$



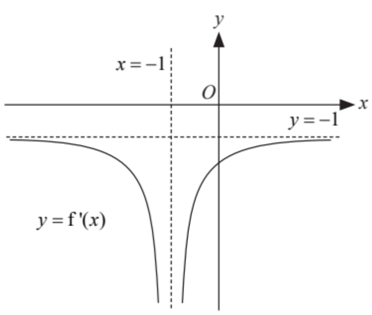
32
Solution

The graph of $y = f'(x)$

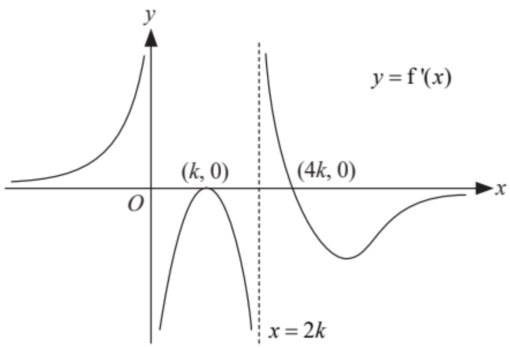


33
Solution

The graph of $y = f'(x)$

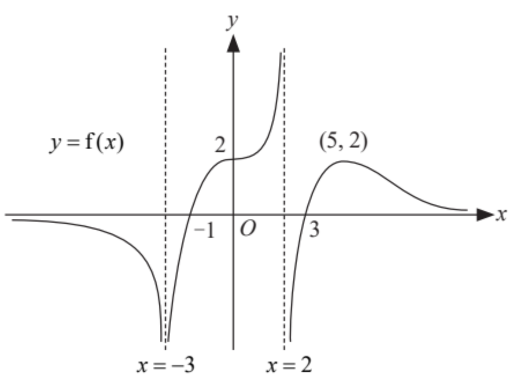


34
Solution

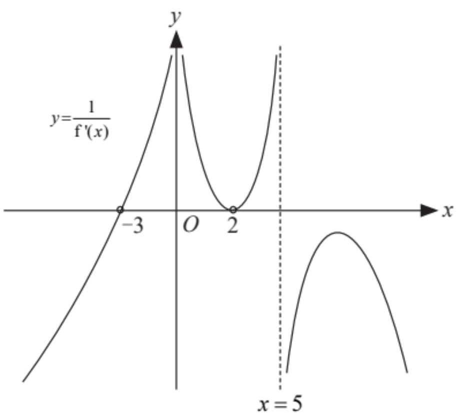


35
Solution

(a) The graph of $y = \mathrm{f}(x)$



(b) The graph of $y = \dfrac{1}{\mathrm{f}'(x)}$



36
Solution

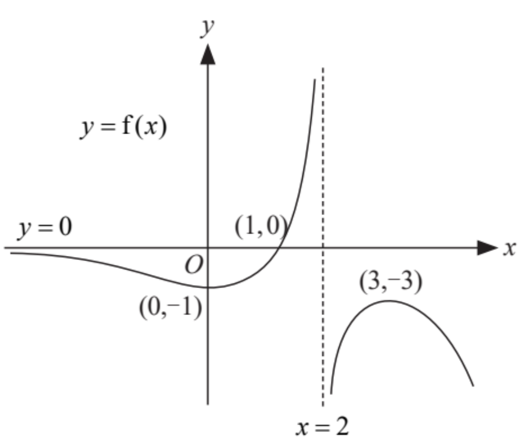
(a) The coordinates $(3,-3)$ is a maximum point.
The coordinates $(0,-1)$ is a minimum point.

Learning point:
Refer to the graph $y = \mathrm{f}'(x)$.
The curve passes through the point $(0,-1)$ from below x-axis to above x-axis. Therefore the point $(0,-1)$ is a minimum point.
The curve passes through the point $(3,-3)$ from above x-axis to belw x-axis. Therefore the point $(3,-3)$ is a maximum point.

(b)(i) The range of values of x where f is decreasing is $x \le 0$ or $x \ge 3$.

(b)(ii) The range of values of x where f is is concave upwards is $-1 \le x < 2$.

(c) The graph of $\mathrm{f}(x)$



Exercise 8

F Mixed Practice

37
Solution

(a) $$y = \frac{4x+9}{x+2}$$
$$= \frac{4(x+2)+1}{x+2}$$
$$= 4 + \frac{1}{x+2}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{(x+2)^2}$$

$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{(x+2)^2} > 0,$ for all real values of x, $x \neq -2$.

Thus, gradient of C is negative for all points on C.

(b) $y = 4 + \frac{1}{x+2}$

Equations of the asymptotes of C are $x = -2$ and $y = 4$.

(c) $y = 4 + \frac{1}{x+2} \xrightarrow{\text{Replace } x \text{ by } (x-2)} y = 4 + \frac{1}{(x-2)+2} = 4 + \frac{1}{x}$

Translate the graph C 2 units in the positive x direction

$y = 4 + \frac{1}{x} \xrightarrow{\text{Replace } y \text{ by } (y+4)} y + 4 = 4 + \frac{1}{x} = \frac{1}{x}$

Translate the resultant graph 4 units in the negative y direction

38
Solution

(a)(i) $y = 2x^{-1} - 3x^{-2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x^{-2} + 6x^{-3}$$
$$= -\frac{2}{x^2} + \frac{6}{x^2}$$
$$= \frac{-2x+6}{x^3}$$

For C is decreasing, $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$.

$$\frac{-2x+6}{x^3} < 0$$
$$\frac{2x-6}{x^3} > 0$$
$$x < 0 \quad \text{or} \quad x > 3$$

$\therefore$ the set of values of x is $\{x \in \mathbb{R},\ x < 0 \text{ or } x > 3\}$

(a)(i) $\frac{\mathrm{d}y}{\mathrm{d}x} = -2x^{-2} + 6x^{-3}$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4x^{-3} - 18x^{-4}$$
$$= \frac{4}{x^3} - \frac{18}{x^4}$$
$$= \frac{4x-18}{x^4}$$

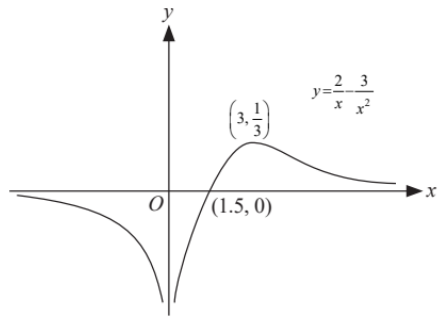
For C concave upwards, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$.

$$\frac{4x-18}{x^4} > 0$$

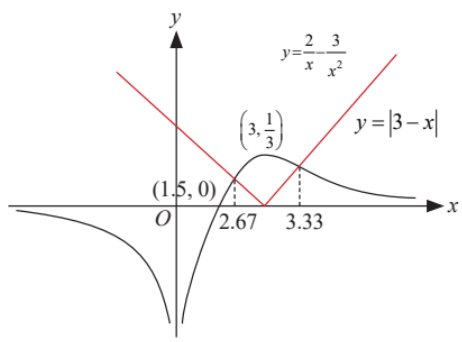
For all real values of x, $x \neq 1$, $x^4 > 0$

$$4x - 18 > 0$$
$$4x > 18$$
$$x > \frac{9}{2}$$

(b) The graph of C of equation $y=\frac{2}{x}-\frac{3}{x^2}$



(c) Add the graph $y=|3-x|$



Use GC to find the points of intersection between $\frac{2x-3}{x^2}$ and $|3-x|$.

$\therefore$ $x=3.33$ and 2.67.

For $\frac{2x-3}{x^2}<|3-x|$, the required inequalities are $x>3.33$ or $x<2.67, x\neq 0$

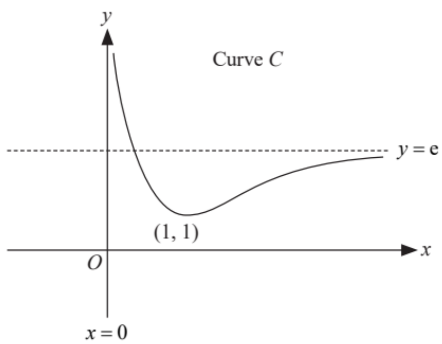
Learning point:

To solve $\frac{2x-3}{x^2}<|3-x|$ graphically. Refer to the diagram and look out the range of values of x such that line $y=|3-x|$ is above the graph $y=\frac{2x-3}{x^2}$.

39
Solution

(a) As $t \to -1$, $x \to \infty$, $y \to e^{(-1)^2} = e$

As $t \to \infty$, $y \to \infty$, $x \to 0$



(b) Given $x = \dfrac{1}{\sqrt{t+1}}$

$= (t+1)^{-\frac{1}{2}}$ .... (1)

and $y = e^{t^2}$ .... (2)

Differentiate (1) with respect to t

$$\frac{dx}{dt} = -\frac{1}{2\sqrt{(t+1)^3}}$$

Differentiate (2) with respect to t

$$\frac{dy}{dt} = 2te^{t^2}$$

Using the Chain Rule,

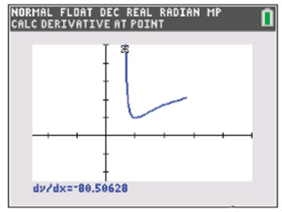
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= (2te^{t^2}) \times \left(-2\sqrt{(t+1)^3}\right)$$

$\therefore \dfrac{dy}{dx} = -4te^{t^2}\sqrt{(t+1)^3}$ (Shown)

(c) Use GC, the gradient of tangent at $t=\dfrac{5}{4}$ is -80.50628



$\therefore$ Gradient of normal at $t=\dfrac{5}{4}$

$$=-\frac{1}{\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=\frac{5}{4}}}$$

$$=-\frac{1}{(-80.50628)}$$

$$=0.012421$$

Substitute $t=\dfrac{5}{4}$ into (1) and (2).

From (1): $x=\left(\dfrac{5}{4}+1\right)^{-\frac{1}{2}}=\dfrac{2}{3}$

From (2): $y=\mathrm{e}^{\left(\frac{5}{4}\right)^2}=\mathrm{e}^{\frac{25}{16}}$

$\therefore$ the point at $t=\dfrac{5}{4}$ is $\left(\dfrac{2}{3},\mathrm{e}^{\frac{25}{16}}\right)$.

Equation of the normal at $t=\dfrac{5}{4}$ (or at the point $\left(\dfrac{2}{3},\mathrm{e}^{\frac{25}{16}}\right)$)

$$y-\mathrm{e}^{\frac{25}{16}}=0.012421\left(x-\frac{2}{3}\right) \text{ .... (3)}$$

The normal at $t=\dfrac{5}{4}$ meets the y-axis at the point P, i.e. $y=0$.

Substitute $y=0$ into (3)

$$y-\mathrm{e}^{\frac{25}{16}}=0.012421\left(0-\frac{2}{3}\right)$$

$$y=4.76245$$

The coordinates P are $(0, 4.76245)$.

Given that the point Q on the curve C has a parameter q,
i.e. $t=q$.
Substitute $t=q$ into (1) and (2).

From (1): $x=(q+1)^{-\frac{1}{2}}$

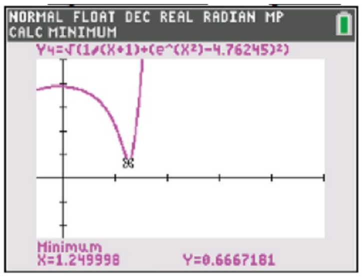
From (2): $y=\mathrm{e}^{(q)^2}=\mathrm{e}^{q^2}$

$\therefore$ the coordinates Q are $\left(\dfrac{1}{\sqrt{q+1}},\mathrm{e}^{q^2}\right)$.

Method 1 (Using Graphical)

Length PQ

$$= \sqrt{\left(0 - \frac{1}{\sqrt{q+1}}\right)^2 + (e^{q^2} - 4.76245)^2} \quad \triangleleft \text{ using distance formulae } = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$



From the graph, minimum PQ occurs when $q = 1.25$.

$\therefore$ the minimum $PQ = 0.667$ units.

Method 2

For PQ to be a minimum, gradient of the line PQ = 0.

$$\therefore \frac{e^{q^2} - 4.7625}{0 - \frac{1}{\sqrt{q+1}}} = 0 \qquad \triangleleft \text{ using gradient formulae } = \frac{y_1 - y_2}{x_1 - x_2}$$

Using GC, $q = 1.25$

Thus Minimum PQ

$$= \sqrt{\left(\frac{1}{\sqrt{1.2493+1}}\right)^2 + \left(e^{\frac{25}{16}} - 4.76245\right)^2}$$

$= 0.667$

$\therefore$ value of q is 1.25. The minimum length PQ is 0.667.

40
Solution

(a) Given $x = \sin t - \cos t$ .... (1)

and $y = \ln\left(\dfrac{t}{\pi}\right)$ .... (2)

Differentiate (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \cos t + \sin t$$

Differentiate (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{t}$$

Using the Chain Rule,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} \\ &= \frac{1}{t} \times \frac{1}{(\cos t + \sin t)} \\ &= \frac{1}{t(\cos t + \sin t)} \quad \text{........................... (3)}\end{aligned}$$

The curve crosses the y-axis at the point P, i.e. $x = 0$.
Substitute $x = 0$ into (1).

From (1) $\sin t - \cos t = 0$

$$\sin t = \cos t$$

$$\tan t = 1$$

$$t = \frac{\pi}{4}$$

Substitute $t = \dfrac{\pi}{4}$ into (3).

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\frac{\pi}{4}\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)} \\ &= \frac{1}{\frac{\pi}{4}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)} \\ &= \frac{1}{\frac{\pi}{4}\left(\frac{2}{\sqrt{2}}\right)} \\ &= \frac{1}{\frac{\pi}{2\sqrt{2}}} = \frac{2\sqrt{2}}{\pi}\end{aligned}$$

Gradient of tangent at $P = \dfrac{2\sqrt{2}}{\pi}$

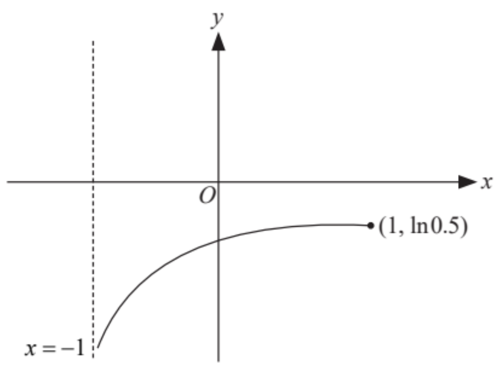
(b) As $t \to 0$, $y = \ln\left(\dfrac{t}{\pi}\right) \to -\infty$

$x = \sin t - \cos t \to \sin 0 - \cos 0 = -1$

$\therefore x \to -1,\ y \to -\infty$

The vertical asymptote is $x = -1$.

(c) The graph of curve C



(d) Refer to the graph in **(c)**.

For the graph to be strictly increasing, the range of x is $-1 < x < 1$

41
Solution

(a)(i) Given $x = e^{3t}$ .... (1)

and $y = t^2$ .... (2)

Differentiate (1) with respect to t

$$\frac{dx}{dt} = 3e^{3t}$$

Differentiate (2) with respect to t

$$\frac{dy}{dt} = 2t$$

Using the Chain Rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{dy}{dt} \times \frac{dt}{dx} \\ &= 2t \times \frac{1}{3e^{3t}} \\ &= \frac{2t}{3e^{3t}} \quad \text{...........................} \ (3)\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{2t}{3e^{3t}}$$

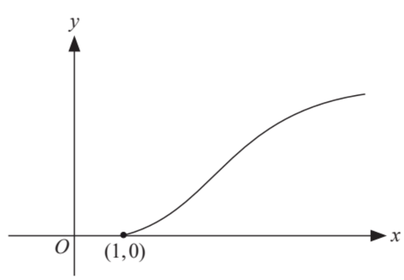
When $\frac{dy}{dx} = 0$,

$$\therefore \quad \frac{2t}{3e^{3t}} = 0$$

$$t = 0$$

$\therefore \ t = 0$

(a)(ii) The curve C



(b)(i) $x^2 - 2xy + 2y^2 = k$ .... (1)

Differentiate (1) with respect to x.

$$2x - 2\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) + 4y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y - x}{2y - x}$$

(b)(ii) For tangents which are parallel to the line $y = x$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$.

$$\frac{y - x}{2y - x} = 1$$

$$y - x = 2y - x$$

$$y = 0$$

Substitute $y = 0$ into (1)

$$x^2 - 2x(0) + 2(0)^2 = k$$

$$x^2 = k$$

Given that there are 2 tangents parallel to the line $y = x$, $\therefore$ the equation $x^2 = k$ is defined.

$\therefore\ k > 0$

(b)(iii) For tangents which are parallel to the y-axis, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined.

$$2y - x = 0$$

$$x = 2y$$

Substitute $x = 2y$ and $k = 4$ into (1)

$$2(y)^2 - 2(2y)y + 2y^2 = 4$$

$$y = \pm\sqrt{2}$$

Substitute $y = \pm\sqrt{2}$ into $x = 2y$

$\therefore\ x = \pm 2\sqrt{2}$

The coordinates are $(-2\sqrt{2}, -\sqrt{2})$ and $(2\sqrt{2}, \sqrt{2})$.

42
Solution

(a) Given $x^2 - xy + y^3 = 16$ .... (1)

Differentiate (1) with respect to x

$$2x - \left(x\frac{dy}{dx} + y\right) + 3y^2\frac{dy}{dx} = 0$$

$$2x - x\frac{dy}{dx} - y + 3y^2\frac{dy}{dx} = 0$$

$$\left(3y^2 - x\right)\frac{dy}{dx} - y + 2x = 0 \quad \text{(Shown)} \text{ .... (2)}$$

(b) Differentiate (2) with respect to x

$$\left(3y^2 - x\right)\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(6y\frac{dy}{dx} - 1\right) - \frac{dy}{dx} + 2 = 0$$

$$\left(3y^2 - x\right)\frac{d^2y}{dx^2} + (6y)\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx} - \frac{dy}{dx} + 2 = 0$$

$$(3y^2 - x)\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} + 2 = 0 \quad \text{(Shown)} \text{ .... (3)}$$

(c) Given that C intersects the positive x-axis at the point N.

Substitute $y = 0$ into (1)

$$x^2 - x(0) + (0)^3 = 16$$

$$x^2 = 16$$

$$x = 4 \text{ or } x = -4 \text{ (rejected since } x > 0)$$

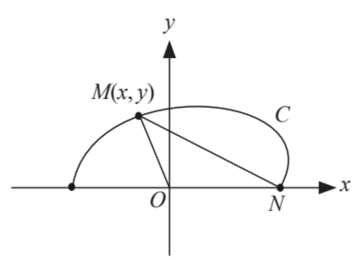
The coordinates N are $N(4, 0)$

Area of ΔOMN

$A = \frac{1}{2}(ON)$(perpendicular distance from M to x-axis)

$= \frac{1}{2}(4)(y)$

$= 2y$ (Shown) .... (4)



(d) Differentiate (3) with respect to x

$$\frac{dA}{dx} = 2\frac{dy}{dx}$$

For stationary value of A, $\frac{dA}{dx} = 0$

i.e. $2\frac{dy}{dx} = 0$ .... (5)

From (1): $\left(3y^2 - x\right)\dfrac{dy}{dx} - y + 2x = 0$

$$\left(3y^2 - x\right)\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{3y^2 - x} \text{ .... (6)}$$

At stationary, $\dfrac{dy}{dx} = 0$

From (6) $\quad \dfrac{2(y-2x)}{3y^2 - x} = 0$

$$2(y - 2x) = 0$$

$$y = 2x \text{ .... (7)}$$

Substitute (7) into (1)

$$x^2 - x(2x) + (2x)^3 = 16$$

$$8x^3 - x^2 - 16 = 0$$

From GC, $x = 1.302996$

$\therefore$ the value of x for which A has a stationary value is 1.30

To find the nature of the stationary point, refer to (3)

$$2 + (3y^2 - x)\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} = 0$$

At stationary, $\dfrac{dy}{dx} = 0$.

$$\therefore \quad 2 + (3y^2 - x)\frac{d^2y}{dx^2} + 6y(0)^2 - 2(0) = 0$$

$$(3y^2 - x)\frac{d^2y}{dx^2} = -2$$

$$\frac{d^2y}{dx^2} = -\frac{2}{3y^2 - x} \text{ .... (8)}$$

Substitute (7) into (8)

$$\frac{d^2y}{dx^2} = -\frac{2}{3(2x)^2 - x}$$

$$= -\frac{2}{12x^2 - x}$$

When $x = 1.30$

$$= -\frac{2}{12x^2 - x}$$

$$= -0.105 < 0$$

$\therefore$ A is a maximum

43
Solution

(a) (i) 1. Reflect graph about the x - axis
2. Scale graph parallel to the y - axis by factor of $(6+a)$
3. Translate graph along / in the negative y - direction by 2 units

$$y=\frac{2x+a}{3-x}$$
$$=-\frac{2x+a}{x-3} \quad \triangleleft \text{ use long division}$$
$$=-2-\frac{6+a}{x-3}$$
$$=-2-(6+a)(x-3)^{-1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=(6+a)(x-3)^{-2}$$
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}=-2(6+a)(x-3)^{-3}$$
$$=-\frac{2(6+a)}{(x-3)^3}$$

For the graph is concave upwards, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}>0.$

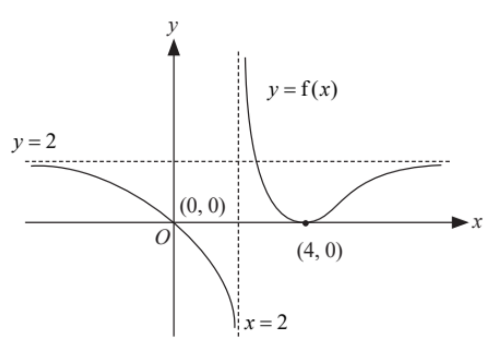
i.e. $-\frac{2(6+a)}{(x-3)^3}>0$

Given $a<-6$, $\therefore\ 6+a<0$

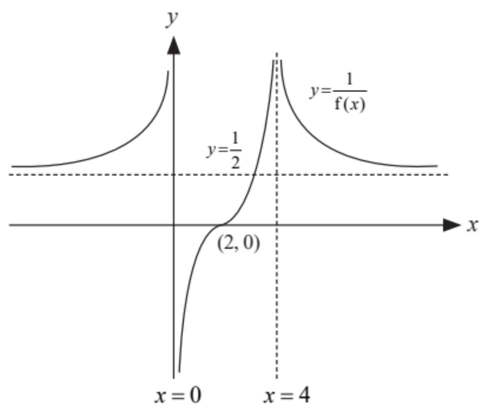
Hence, $(x-3)^3>0$
$x>3$

The range of values of x where the graph of $y=\frac{2x+a}{3-x}$ is concave upwards is $x>3$.

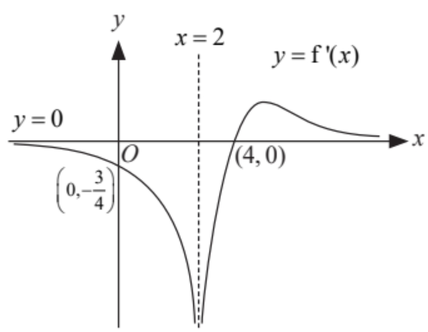
(b)(i) The graph of $y=\mathrm{f}(x)$



(b)(ii) The graph of $y = \dfrac{1}{f(x)}$



(b)(iii) The graph of $y = f'(x)$



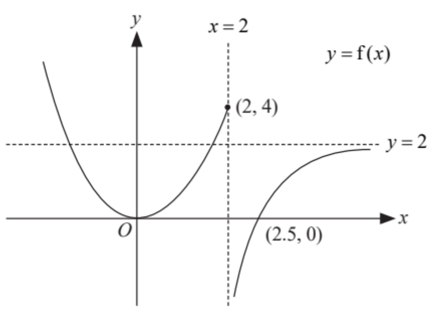
44
Solution

(a) Scale parallel to x - axis by a factor of $\frac{1}{2}$.

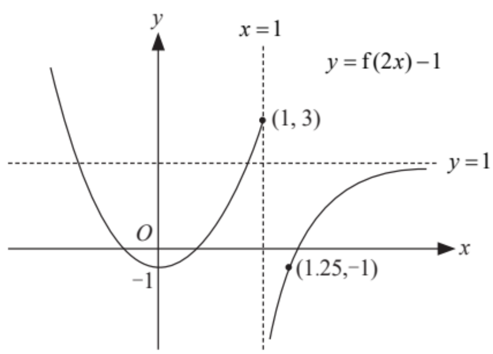
Translate 1 unit in the negative y-direction.

(OR in reverse sequence)

(b) The graph of $y = \mathrm{f}(x)$



The graph of $y = \mathrm{f}(2x) - 1$



(c) (Any of the following 2 reasons)

For the graph of $y = \mathrm{f}'(x)$,

1. there should be a horizontal asymptote $y = 0$ instead of $y = 2$.
2. it should end at $(2, 4)$ for the segment for which $x \le 2$ since $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=2} = 2(2) = 4$.
3. the segment for which $x \le 2$ should be a straight line (with equation $y = 2x$).

Exercise 8

G Higher Order Questions

45
Solution

Given $y = \dfrac{\lg x}{x}$ ◁ use change base law

$$= \frac{1}{x}\left(\frac{\ln x}{\ln 10}\right)$$

$$= \frac{1}{\ln 10}\left(\frac{\ln x}{x}\right)$$

Differentiate both sides with respect to x

$$\frac{dy}{dx} = \frac{1}{\ln 10}\left[\frac{x\left(\frac{1}{x}\right) - \ln x}{x^2}\right]$$

$$= \frac{1}{\ln 10}\left[\frac{1 - \ln x}{x^2}\right]$$

For stationary points, $\dfrac{dy}{dx} = 0.$

i.e. $$\frac{1}{\ln 10}\left[\frac{1 - \ln x}{x^2}\right] = 0$$

$$1 - \ln x = 0$$

$$x = e$$

Given that the stationary point of the curve is a maximum

$\therefore$ the maximum point at $x = e$

Since $x = e$ gives the only maximum point,

$$y\big|_{x=e} > y\big|_{x=\pi}$$

$$\frac{\lg e}{e} > \frac{\lg \pi}{\pi}$$

$$\pi \lg e > e \lg \pi$$

$$\lg e^{\pi} > \lg \pi^{e}$$

$\therefore$ $e^{\pi} > \pi^{e}$ (Deduced)

46
Solution

Given $x^2+3y^2+2xy=3-k$ .... (1)

Differentiate (1) with respect to x

$2x+6y\dfrac{dy}{dx}+2x\dfrac{dy}{dx}+2y=0$ .... (2)

For tangent parallel to x-axis, $\dfrac{dy}{dx}=0$.

From (2): $2x+0+0+2y=0$ ◁ substitute $\dfrac{dy}{dx}=0$ into (2)

$\therefore$ $y=-x$ .... (3)

Putting (3) into (1)

$$x^2+3(-x)^2+2x(-x)=3-k$$
$$2x^2=3-k$$

There is no real solution for x when $\dfrac{3-k}{2}<0$.

Hence,
$$\frac{3-k}{2}<0$$
$$3-k<0$$
$$3<k$$

$\therefore$ C does not have any stationary points if $k>3$.

47
Solution

(a) Given $kxe^{y}+ke^{x}=y^{2}+k^{2}$

Differentiating with respect to x

$$ke^{y}+kxe^{y}\frac{\mathrm{d}y}{\mathrm{d}x}+ke^{x}=2y\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}(2y-kxe^{y})=ke^{x}+ke^{y}$$

$$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{ke^{x}+ke^{y}}{2y-kxe^{y}}$$

(a) When the tangent to the curve to be parallel to the x - axis, gradient is 0.

i.e. $\frac{\mathrm{d}y}{\mathrm{d}x}=0$

$\therefore$ $ke^{x}+ke^{y}=0$

For all real values of x, y, $e^{x}>0$, $e^{y}>0$

Hence, $ke^{x}+ke^{y}>0$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x}\neq 0$ for all $x, y\in\mathbb{R}$

There is no point where the tangent is parallel to the x - axis.

48
Solution

(a) Given $x = \ln(\cos 2\theta)$ .... (1)

and $y = \ln(\sin 2\theta)$ .... (2)

Differentiate (1) with respect to θ

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}\theta} &= \frac{-2\sin 2\theta}{\cos 2\theta}\\ &= -2\tan 2\theta\end{aligned}$$

Differentiate (2) with respect to θ

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}\theta} &= \frac{2\cos 2\theta}{\sin 2\theta}\\ &= \frac{2}{\tan 2\theta}\end{aligned}$$

Using the Chain Rule,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}}\\ &= \frac{\mathrm{d}y}{\mathrm{d}\theta} \times \frac{\mathrm{d}\theta}{\mathrm{d}x}\\ &= \frac{2}{\tan 2\theta} \times \left(\frac{1}{-2\tan 2\theta}\right)\\ &= -\frac{1}{\tan^2 2\theta} \quad .... (3)\end{aligned}$$

Given $0 < \theta < \frac{\pi}{4}$

$0 < 2\theta < \frac{\pi}{2}$ ◁ multiply 2 all sides

$\therefore \tan 2\theta > 0$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\tan^2 2\theta} < 0$

Hence gradient of C is always negative for all points on C.

(b) When $\theta \to 0$, $\tan^2 2\theta \to 0$.

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\tan^2 2\theta} \to -\infty$.

Hence tangent is parallel to y - axis (or $x = 0$).

When $\theta \to \frac{\pi}{4}$, $\tan^2 2\theta \to \infty$.

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\tan^2 2\theta} \to 0$.

Hence tangent is parallel to x - axis (or $y = 0$).

(c) Given $\theta = \dfrac{\pi}{8}$, substitute $\theta = \dfrac{\pi}{8}$ into (1), (2) and (3).

From (1): $x = \ln\left(\dfrac{1}{\sqrt{2}}\right)$

$= -\ln\sqrt{2}$

$= -\dfrac{1}{2}\ln 2$

From (2): $y = \ln\left(\dfrac{1}{\sqrt{2}}\right)$

$= -\dfrac{1}{2}\ln 2$

From (3): $\dfrac{dy}{dx} = -\dfrac{1}{\tan^2 \dfrac{\pi}{4}}$

$= -1$

Equation of tangent at $\theta = \dfrac{\pi}{8}$

$$y - \left(-\frac{1}{2}\ln 2\right) = -\left[x - \left(-\frac{1}{2}\ln 2\right)\right] \qquad \triangleleft \text{ using } y - y_1 = m(x - x_1)$$

$$y + \frac{1}{2}\ln 2 = -x - \frac{1}{2}\ln 2$$

Hence $y + x + \ln 2 = 0$ (Shown) .... (3)

(c) Substitute (1) and (2) into (3)

$\ln(\sin 2\theta) + \ln(\cos 2\theta) + \ln 2 = 0$

$\ln(2\sin 2\theta\cos 2\theta) = 0$

$\ln(\sin 4\theta) = 0$

$\sin 4\theta = e^0$

$\therefore\ 4\theta = \dfrac{\pi}{2}$

$\theta = \dfrac{\pi}{8}$ where $0 < \theta < \dfrac{\pi}{4}$

Since there is only 1 value of θ, the tangent meets C only once at $\theta = \dfrac{\pi}{8}$.

$\therefore$ the tangent will not meet C again.

49
Solution

(a) Given $x = 1\frac{t}{\sqrt{1-4t^2}}$ .... (1)

and $y = \sin^{-1} 2t$ .... (2)

Differentiate (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{(1-4t^2)^{\frac{1}{2}} - t\left(\frac{1}{2}\right)(1-4t^2)^{-\frac{1}{2}}(-8t)}{(1-4t^2)}$$

$$= \frac{(1-4t^2)^{-\frac{1}{2}}\left[(1-4t^2)+4t^2\right]}{(1-4t^2)}$$

$$= (1-4t^2)^{-\frac{3}{2}}$$

Differentiate (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{2}{\sqrt{1-4t^2}}$$

Using the Chain Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

$$= \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$$

$$= \frac{2}{\sqrt{1-4t^2}} \times \frac{1}{(1-4t^2)^{-\frac{3}{2}}}$$

$$= \frac{2}{\sqrt{1-4t^2}} \times (1-4t^2)^{\frac{3}{2}}$$

$= 2(1-4t^2)$ (Shown) .... (3)

Given $-\frac{1}{2} < t < \frac{1}{2}$,

$\therefore \quad t^2 < \frac{1}{4}$

$4t^2 < 1$

i.e. $1-4t^2 > 0$

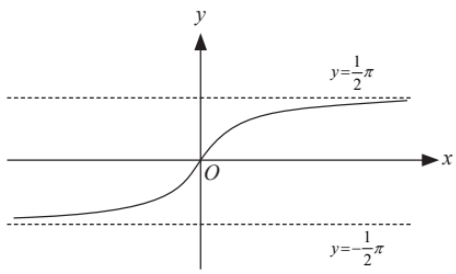
$\frac{\mathrm{d}y}{\mathrm{d}x} = 2(1-4t^2)$ is always positive, for $-\frac{1}{2} < t < \frac{1}{2}$

Thus gradient is positive for all points on the curve.

(b) As $x \to \pm\infty, t \to \pm\frac{1}{2}, \therefore\ y \to \pm\frac{\pi}{2}$

The curve approaches $y = \pm\frac{\pi}{2}$.

The curve C



(c) Given that at point A, the parameter $t = \frac{1}{2\sqrt{2}}$.

Substitute $t = \frac{1}{2\sqrt{2}}$ into (1), (2) and (3).

From (1): $x = \dfrac{\frac{1}{2\sqrt{2}}}{\sqrt{1-4\left(\frac{1}{8}\right)}}$

$= \frac{1}{2}$

From (2): $y = \sin^{-1} 2\left(\frac{1}{2\sqrt{2}}\right)$

$= \frac{\pi}{4}$

From (3): $\frac{dy}{dx} = 2\left(1-4\left(\frac{1}{2\sqrt{2}}\right)^2\right)$

$= 1$

Equation of tangent l:

$$y - \frac{\pi}{4} = 2\left(1-4\left(\frac{1}{8}\right)\right)\left(x - \frac{1}{2}\right)$$

$$y = x - \frac{1}{2} + \frac{\pi}{4} \quad \text{.................................... (3)}$$

(c) Substitute (1) and (2) into (3)

$$\sin^{-1} 2t = \frac{t}{\sqrt{1-4t^2}} - \frac{1}{2} + \frac{\pi}{4}$$

From GC, $t = 0.35355$ or $t = -0.47564$

Now, $t = 0.35355$ corresponds to point A, therefore the tangent will cut curve C again. (Shown)

Substitute $t = -0.47564$ into (1) and (2)

From (1) $x = \dfrac{-0.47564}{\sqrt{1-4(-0.47564)^2}} = -1.54278,$

From (2) $y = \sin^{-1} 2(-0.47564) = -1.25737$

Coordinates of B are $(-1.54, -1.26)$.

50
Solution

(a) Given $x=\dfrac{k}{t},\ \ y=t^2+t,\ \ t\in\mathbb{R}, k>0$

As $x\to\pm\infty, t\to 0, y\to 0.$

$\therefore y=0$ is a horizontal asymptote.

As $x\to 0, t\to\pm\infty, y\to\infty.$

$\therefore x=0$ is a vertical asymptote.

Learning point:

An asymptote is a line which the graph tends towards at the extreme ends. Thus to explain whether a line is an asymptote, we will need to check the extreme ends (i.e. $x\to\pm\infty$ & $y\to\infty$.)

(b) Given $x=\dfrac{k}{t}$ .... (1)

and $\quad y=t^2+t$ .... (2)

Differentiate (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{k}{t^2}$$

Differentiate (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t}=2t+1$$

Using the Chain Rule,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}\\&=\frac{2t+1}{-\frac{k}{t^2}}\\&=-\frac{1}{k}t^2(2t+1)\end{aligned}$$

Given that there is one minimum point, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$

i.e. $\quad -\dfrac{1}{k}t^2(2t+1)=0$

$$2t+1=0 \quad \text{where } t\neq 0$$

$$t=-\frac{1}{2}$$

$x=\dfrac{k}{-\frac{1}{2}}=-2t,\ y=\left(-\dfrac{1}{2}\right)^2+\left(-\dfrac{1}{2}\right)=-\dfrac{1}{4}$

Coordinates of minimum point at $t=-\dfrac{1}{2}$: $\left(-2k,-\dfrac{1}{4}\right)$

(c) When the graph C cuts x-axis, $y = 0$.

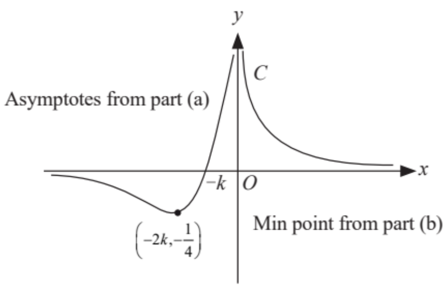
From (1), let $\quad t(t+1) = 0$

$$t = -1 \quad (\because t \neq 0)$$

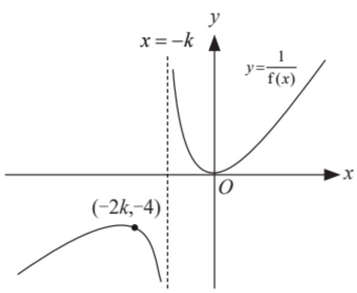
Substitute $t = -1$ into (2)

$\therefore\ x = -k$

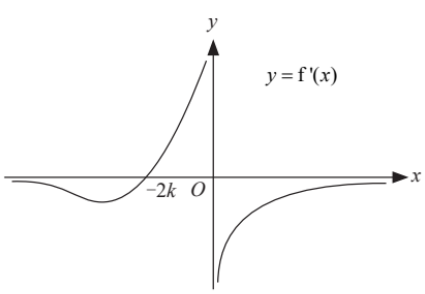
The coordinates are $(-k, 0)$



(d) The graph of $y = \dfrac{1}{\mathrm{f}(x)}$



(e) The graph of $y = \mathrm{f}'(x)$



51
Solution

(a) Given $x = 3\sin\theta\cos^2\theta$ .... (1)

and $y = \cos^3\theta$ .... (2)

Differentiate (1) with respect to θ

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}\theta} &= 3\sin\theta(-2\cos\theta\sin\theta) + 3\cos^2\theta\cos\theta \\ &= 3\cos^3\theta - 6\sin^2\theta\cos\theta\end{aligned}$$

Differentiate (2) with respect to θ

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}\theta} &= 3\cos^2\theta(-\sin\theta) \\ &= -3\sin\theta\cos^2\theta\end{aligned}$$

Using the Chain Rule,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} &= \frac{-3\sin\theta\cos^2\theta}{3\cos^3\theta - 6\sin^2\theta\cos\theta} \\ &= \frac{\sin\theta\cos\theta}{2\sin^2\theta - \cos^2\theta} \quad \text{(Shown)}\end{aligned}$$

(b) Given that tangent to the cuvre is parallel to the line $y = 2x + 3$,

i.e. $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$

$$\frac{\sin\theta\cos\theta}{2\sin^2\theta - \cos^2\theta} = 2$$

$$\sin\theta\cos\theta = 2(2\sin^2\theta - \cos^2\theta)$$

$$4\sin^2\theta - 2\cos^2\theta - \sin\theta\cos\theta = 0$$

Using GC, $\theta = 0.70,\ 2.61$

(c) Equation of normal at $(3\sin a\cos^2 a, \cos^3 a)$ is

$$y - \cos^3 a = -\frac{2\sin^2 a - \cos^2 a}{\sin a\cos a}(x - 3\sin a\cos^2 a) \text{ ....(3)}$$

Given that the normals to the curve C at A and B intersect at the origin, i.e. $x = 0$ and $y = 0$.

Substitute $x = 0$ and $y = 0$ into (3)

$$0 - \cos^3\theta = -\frac{2\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}(0 - 3\sin a\cos^2\theta)$$

$$-\cos^3\theta = -\frac{2\sin^2 a - \cos^2\theta}{\sin\theta\cos\theta}(-3\sin\theta\cos^2\theta)$$

$$\cos^2\theta = 3\cos^2\theta - 6\sin^2\theta$$

$$6\sin^2\theta = 2\cos^2\theta$$

$$\frac{\sin^2\theta}{\cos^2\theta} = \frac{2}{6}$$

$$\tan^2\theta = \frac{1}{3}$$

$$\tan\theta = \pm\frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

Since $a < b$,

$$a = \frac{\pi}{6}, b = \frac{5\pi}{6}$$

52
Solution

(a) Given $y=(x-y)^2$

Differentiate both sides with respect to x

$$\frac{\mathrm{d}y}{\mathrm{d}x}=2(x-y)\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

Differentiate both sides with respect to x

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)+2(x-y)\left(-\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\left(\frac{\frac{\mathrm{d}y}{\mathrm{d}x}}{1-\frac{\mathrm{d}y}{\mathrm{d}x}}\right)\left(-\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)$$

$$\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 \quad \text{(Shown) .... (1)}$$

(b) If C has a stationary point of inflexion, then both $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=0$.

Subst $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ into (1).

LHS = 0

RHS = $2(1-0)^3=2$

Since LHS $\neq$ RHS for all $x\in\mathbb{R}$, C cannot have any stationary points of inflexion.

53
Solution

(a) If the distance AB is the least, the line segment AB is perpendicular to l.

(b) Given $B(b,\ 2\sqrt{b})$ and $A(a,\ 2a+1)$

Gradient of $BA = \dfrac{2\sqrt{b}-2a-1}{b-a}$

From the equation of the line l, gradient of l is 2.

From **(a)**, the line segment AB is perpendicular to l.

$$\therefore \quad \frac{2\sqrt{b}-2a-1}{b-a} = -\frac{1}{2}$$

$$4\sqrt{b}-4a-2 = a-b \ \text{........................} \ (1)$$

$$a = \frac{1}{5}(b+4\sqrt{b}-2) \ \text{........................} \ (2)$$

(c) Length AB

$= \sqrt{(2\sqrt{b}-2a-1)^2+(b-a)^2}$

$= \sqrt{(2\sqrt{b}-2a-1)^2+(4\sqrt{b}-4a-2)^2}$ $\quad \triangleleft$ replace $b-a = 4\sqrt{b}-4a-2,$ refer to (1)

$= \sqrt{(2\sqrt{b}-2a-1)^2+2^2(\sqrt{b}-a-1)^2}$

$= \sqrt{5(2\sqrt{b}-2a-1)^2}$

$= \sqrt{5\left(2\sqrt{b}-\dfrac{2}{5}(b+4\sqrt{b}-2)-1\right)^2}$ $\quad \triangleleft$ replace $a = \dfrac{1}{5}(b+4\sqrt{b}-2),$ refer to (2)

$$\therefore \ (AB)^2 = 5\left(\frac{2}{5}\sqrt{b}-\frac{2}{5}b-\frac{1}{5}\right)^2$$

$$= \frac{1}{5}(2\sqrt{b}-2b-1)^2$$

$$= \frac{1}{5}(2b-2\sqrt{b}+1)^2 \ \text{(Deduced), where } m = \frac{1}{5} \ \text{......................} \ (3)$$

Differentiate (3) both sides with respect to b

$$2AB\frac{\mathrm{d}AB}{\mathrm{d}b}=\frac{2}{5}(2b-2\sqrt{b}+1)\left(2-\frac{1}{\sqrt{b}}\right)$$

When $\frac{\mathrm{d}AB}{\mathrm{d}b}=0,$

$$\frac{2}{5}(2b-2\sqrt{b}+1)\left(2-\frac{1}{\sqrt{b}}\right)=0$$

Since $(-2)^2-4(2)(1)<0$, $(2b-2\sqrt{b}+1)=0$ has no real solution.

Consider $(2b-2\sqrt{b}+1)=0$

$$2-\frac{1}{\sqrt{b}}=0$$

$$b=\frac{1}{4}$$

Substitute $b=\frac{1}{4}$ into $B(b,\,2\sqrt{b})$

$\therefore$ the coordinates of the point on C that is nearest to $l=\left(\frac{1}{4},1\right)$.

(c) Alternative Method I

$$\begin{aligned}(AB)^2&=\frac{1}{5}(2b-2\sqrt{b}+1)^2\\&=\frac{4}{5}\left(b-\sqrt{b}+\frac{1}{2}\right)^2\\&=\frac{4}{5}\left(\left(\sqrt{b}-\frac{1}{2}\right)^2-\frac{1}{4}+\frac{1}{2}\right)^2\\&=\frac{4}{5}\left(\left(\sqrt{b}-\frac{1}{2}\right)^2+\frac{1}{4}\right)^2\end{aligned}$$

Since $\left(\sqrt{b}-\frac{1}{2}\right)^2\geq 0$ for all real b, $(AB)^2$ is the least when $\sqrt{b}=\frac{1}{2}$, that is , $b=\frac{1}{4}$.

Hence the point on C nearest to l is $(b,\,2\sqrt{b})=\left(\frac{1}{4},1\right)$

Alternative Method II

When $(AB)^2$ is the least, tangent to C at B is parallel to l, i.e. gradient of tangent to $C = 2$

$$y^2 = 4x$$

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{y}$$

At $(b, 2\sqrt{b})$, substitute $y = 2\sqrt{b}$ into $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{y}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{2\sqrt{b}}$$

$$= \frac{1}{\sqrt{b}}$$

Given gradient of tangent to $C = 2$

$$2 = \frac{1}{\sqrt{b}}$$

$$b = \frac{1}{4}$$

$\therefore$ the coordinates on C nearest to l is $\left(\frac{1}{4}, 1\right)$

54
Solution

(a) Given $x = \tan\theta$ .... (1)

and $y = \sec\theta$ .... (2)

Differentiate (1) with respect to θ

$$\frac{dx}{d\theta} = \sec^2\theta$$

Differentiate (2) with respect to θ

$$\frac{dy}{d\theta} = \sec\theta\tan\theta$$

Using the Chain Rule,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sec\theta\tan\theta}{\sec^2\theta}$$

$$= \frac{\tan\theta}{\sec\theta}$$

$$= \sin\theta$$

$\therefore$ the gradient function is $\sin\theta$

(b) Equation of tangent at $P(\tan\theta, \sec\theta)$ is

$y - \sec\theta = \sin\theta(x - \tan\theta)$ .... (3)

The tangent at $P(\tan\theta, \sec\theta)$ meets the x-axis at Q.

Substitute $y = 0$ into (3)

$0 - \sec\theta = \sin\theta(x - \tan\theta)$.

$$x = \frac{\sin\theta}{\cos\theta} - \frac{1}{\sin\theta\cos\theta}$$

$$= \frac{\sin^2\theta - 1}{\sin\theta\cos\theta}$$

$$= -\frac{\cos^2\theta}{\sin\theta\cos\theta}$$

$$x = -\cot\theta$$

Tangent at P intersects x-axis at $Q(-\cos\theta, 0)$

Gradient of normal at $P = -\dfrac{1}{\sin\theta}$

Equation of normal at $P(\tan\theta, \sec\theta)$ is

$$y - \sec\theta = -\frac{1}{\sin\theta}(x - \tan\theta)$$

The normal at $P(\tan\theta, \sec\theta)$ meets the x-axis at R

Substitute $y = 0$ into (3)

$$0 - \sec\theta = \frac{-1}{\sin\theta}(x - \tan\theta)$$

$$x = \frac{\sin\theta}{\cos\theta} + \tan\theta$$

$$= 2\tan\theta$$

Normal at P intersects x-axis at $R(2\tan\theta, 0)$

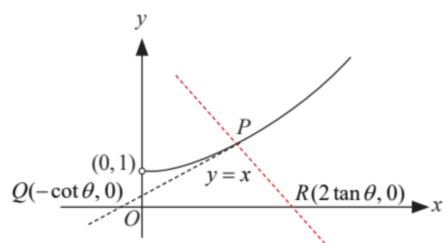
Triangle PQR is a right-angled triangle in circle, so QR is a diameter.

$$A = \pi\left(\frac{QR}{2}\right)^2$$

$$= \pi\left(\frac{2\tan\theta - (-\cot\theta)}{2}\right)^2$$

$$= \pi\left(\tan\theta + \frac{1/\tan\theta}{2}\right)^2$$

$$A = \pi\left(\tan\theta + \frac{1}{2\tan\theta}\right)^2 \quad \text{(Shown)}$$



(c) $\frac{\mathrm{d}}{\mathrm{d}t}\left(t + \frac{1}{2t}\right)^2$

$$= 2\left(t + \frac{1}{2t}\right)\left(1 - \frac{1}{2t^2}\right)$$

At minimum, $2\left(t + \frac{1}{2t}\right)\left(1 - \frac{1}{2t^2}\right) = 0$

$$\left(1 - \frac{1}{2t^2}\right) = 0 \quad \text{or} \quad \left(t + \frac{1}{2t}\right) = 0$$

$$2t^2 = 1 \qquad t + \frac{1}{2t} > 0, \text{ for } t > 0. \therefore \text{ there is no solution}$$

$$t = \frac{1}{\sqrt{2}}, \text{ where } t > 0$$

From **(c)**, $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(t+\dfrac{1}{2t}\right)^2$

$$= 2\left(t+\frac{1}{2t}\right)\left(1-\frac{1}{2t^2}\right)$$

$$= 2\left(t-\frac{1}{4t^3}\right)$$

Differentiate $2\left(t+\dfrac{1}{4t^3}\right)$ with respect to t

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(t+\frac{1}{2t}\right)^2$$

$$= 2\left(1+\frac{3}{4t^4}\right) > 0 \text{ for all } t > 0$$

Using 2nd derivative test

When $t = \dfrac{1}{\sqrt{2}}$, $\left.\dfrac{\mathrm{d}^2}{\mathrm{d}t^2}\left(t+\dfrac{1}{2t}\right)^2\right|_{t=\frac{1}{\sqrt{2}}} = 2\left(1+\dfrac{3}{4\left(\frac{1}{\sqrt{2}}\right)^4}\right) = 8 > 0$ (min)

Substitute $t = \dfrac{1}{\sqrt{2}}$ into $\left(t+\dfrac{1}{2t}\right)^2$.

$\therefore$ the minimum value of $\left(t+\dfrac{1}{2t}\right)^2$ is $\left(\dfrac{1}{\sqrt{2}}+\dfrac{\sqrt{2}}{2}\right)^2 = 2$

Using 1st derivative test

From **(c)**, $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(t+\dfrac{1}{2t}\right)^2 = 2\left(t+\dfrac{1}{2t}\right)\left(1-\dfrac{1}{2t^2}\right)$

$$= 2\left(\frac{2t^2+1}{2t}\right)\left(1-\frac{1}{2t^2}\right)$$

When $t = \left(\dfrac{1}{\sqrt{2}}\right)^-$, then $2t^2 < 1$, so $\left(\dfrac{2t^2-1}{2t^2}\right) < 0$. Also, $\left(\dfrac{2t^2+1}{2t}\right) > 0$ for positive t.

$\therefore$ $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(t+\dfrac{1}{2t}\right)^2 < 0$

When $t = \left(\dfrac{1}{\sqrt{2}}\right)^+$ then $2t^2 > 1$, so $\left(\dfrac{2t^2-1}{2t^2}\right) > 0$. Also, $\left(\dfrac{2t^2+1}{2t}\right) > 0$ for positive t.

$\therefore$ $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(t+\dfrac{1}{2t}\right)^2 > 0$

t	$\left(\frac{1}{\sqrt{2}}\right)^-$	$\frac{1}{\sqrt{2}}$	$\left(\frac{1}{\sqrt{2}}\right)^+$
$\frac{\mathrm{d}}{\mathrm{d}t}\left(t+\frac{1}{2t}\right)^2$	Negative	0	Positive
Tangent	\	—	/

$\therefore \left(t+\frac{1}{2t}\right)^2$ is a minimum when $t=\left(\frac{1}{\sqrt{2}}\right)$.

Substitute $t=\frac{1}{\sqrt{2}}$ into $\left(t+\frac{1}{2t}\right)^2$.

$\therefore$ the minimum value of $\left(t+\frac{1}{2t}\right)^2$ is $\left(\frac{1}{\sqrt{2}}+\frac{\sqrt{2}}{2}\right)^2=2$

Given $0<\theta<\frac{\pi}{2}$, $\tan\theta>0$

So minimizing $A=\pi\left(\tan\theta+\frac{1}{2\tan\theta}\right)^2$ over $0<\theta<\frac{\pi}{2}$ is equivalent to minimizing $A=\pi\left(t+\frac{1}{2t}\right)^2$ for $t>0$.

When $\quad t=\frac{1}{\sqrt{2}}$

$$\tan\theta=\frac{1}{\sqrt{2}}$$

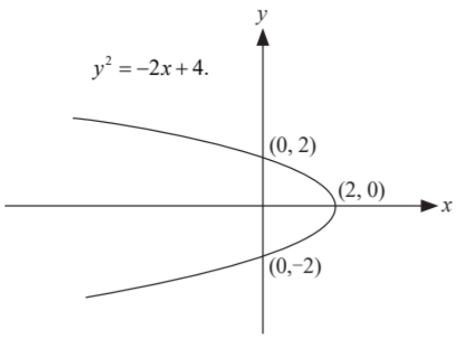
$$\therefore A=\pi\left(\frac{1}{\sqrt{2}}+\frac{1}{2\left(\frac{1}{\sqrt{2}}\right)}\right)^2$$

$$=2\pi$$

Hence the minimum value of $A=2\pi$ when $\theta=\tan^{-1}\frac{1}{\sqrt{2}}$. (Deduced)

55
Solution

(a) The graph of mirror C



(b) Given $y^2 = -2x + 4$

Differentiate (1) with respect to x

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = -2$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{y}$$

At $P\left(-\frac{1}{2}k^2 + 2, k\right)$, the gradient of tangent at P, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{k}$

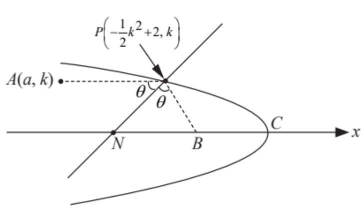
$\therefore$ the gradient of normal to C at $P = k$

Equation of normal to C at P is

$$y - k = k\left[x - \left(-\frac{1}{2}k^2 + 2\right)\right]$$

$$y = kx + \frac{1}{2}k^3 - k \text{ .... (1)}$$

(b) Let N be the x-intercept of the normal to C at P. The coordinates of $B = (b, 0)$



Let $\angle APN = \theta$

Given P bisects angle APB, $\therefore$ PN bisects angle APB.

$\therefore$ $\angle NPB = \theta$

Then $\angle PNB = \theta$ (AP // x - axis, alternate angles)

Hence $BN = BP$ (isosceles triangle)

Substitute $y = 0$ into (1) to find the x-coordinate of N

$$0 = kx + \frac{1}{2}k^3 - k$$

$$0 = x + \frac{1}{2}k^2 - 1$$

$$\therefore \quad x = 1 - \frac{1}{2}k^2$$

The coordinates $N = \left(1 - \frac{1}{2}k^2, 0\right)$

Let the coordinates of $B = (b, 0)$.

Length BP

$$= \sqrt{\left(b - \left(-\frac{1}{2}k^2 + 2\right)\right)^2 + (0-k)^2}$$

Length BN

$$= \sqrt{\left(b - \left(1 - \frac{1}{2}k^2\right)\right)^2}$$

Refer to the diagram. ΔNBP is an isosceles triangle, so $BN = BP$.

i.e. $\sqrt{\left(b - \left(-\frac{1}{2}k^2 + 2\right)\right)^2 + (0-k)^2} = \sqrt{\left(b - \left(1 - \frac{1}{2}k^2\right)\right)^2}$

$$\left(b - \left(-\frac{1}{2}k^2 + 2\right)\right)^2 + k^2 = \left(b - \left(1 - \frac{1}{2}k^2\right)\right)^2$$

$$\left(b - \left(1 - \frac{1}{2}k^2\right)\right)^2 - \left(b - \left(-\frac{1}{2}k^2 + 2\right)\right)^2 = k^2 \qquad \triangleleft \text{ use } a^2 - b^2 = (a+b)(a-b)$$

$$\left(b - \left(1 - \frac{1}{2}k^2\right) + b - \left(-\frac{1}{2}k^2 + 2\right)\right)\left(b - \left(1 - \frac{1}{2}k^2\right) - b + \left(-\frac{1}{2}k^2 + 2\right)\right) = k^2$$

$$2b - 3 + k^2 = k^2$$

$$2b - 3 = 0$$

$$b = \frac{3}{2}$$

Hence, x-coordinate of B is $\frac{3}{2}$

The x-coordinate of B is independent of k.

56
Solution

(a)(i) Given $f(x) = x^2 e^{x^2}$ .... (1)

Differentiate (1) with respect to x

$$f'(x) = x^2(2xe^{x^2}) + 2xe^{x^2} \text{ .... (2)}$$
$$= 2xe^{x^2}(x^2+1)$$

For the function to be increasing, $f'(x) \geq 0$

$\therefore\ 2xe^{x^2}(x^2+1) \geq 0$

Since $x^2+1>0$ and $e^{x^2}>0$, for all $x \in \mathbb{R}$,

$\therefore\ x \geq 0$

The range of values of x for which y is increasing is $x \geq 0$.

(a)(ii) When $x=1$, substitute $x=1$ into (1) and (2).

From (1): $f(1) = e$

From (2): $f'(1) = 4e$

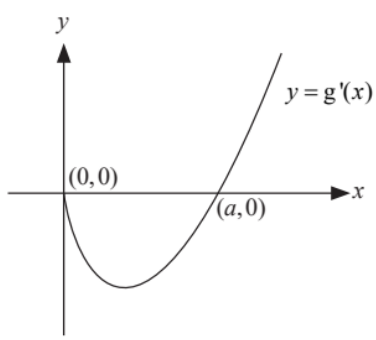
Equation of tangent at $(1, 4e)$ is

$$y - e = 4e(x-1)$$
$$y - e = 4ex - 4e$$

i.e. $\quad y = 4ex - 3e$

The equation of the tangent to the curve, at $x=1$ is $y = 4ex - 3e$.

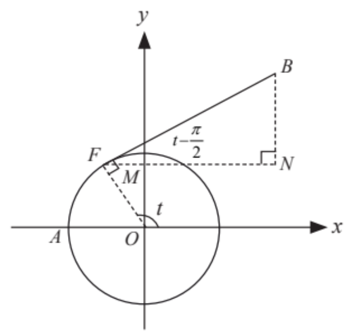
(b) The graph of $y = g'(x)$.



57
Solution

(a) Point F lies on the circumference of the circle,

$\therefore$ the coordinates of point F are $(\cos t, \sin t)$



$\angle AOF = \pi - t$

Arc length $AF = (1)(\pi - t)$ $\quad \triangleleft$ arc length $= r\theta$

$= (1)(\pi - t)$

Length $FB = AB - AF$

$= \pi - (\pi - t)$

$= t$

$\angle NFO = \pi - t$ (int. angle, FN is parallel to x-axis)

$\angle BFN = \angle BNO - \angle NFO$

$= \dfrac{\pi}{2} - (\pi - t)$

$= t - \dfrac{\pi}{2}$

$\therefore$ angle that FB makes with the horizontal $= t - \dfrac{\pi}{2}$.

x-coordinate of B

$= \cos t + t\cos\left(t - \dfrac{\pi}{2}\right)$ $\quad \triangleleft \cos(-\theta) = \cos\theta$

$= \cos t + t\cos\left(\dfrac{\pi}{2} - t\right)$ $\quad \triangleleft \cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta$

$= \cos t + t\sin t$

y-coordinate of B

$= \sin t + t\sin\left(t - \dfrac{\pi}{2}\right)$

$= \sin t - t\sin\left(\dfrac{\pi}{2} - t\right)$

$= \sin t - t\cos t$

$\therefore$ the parametric equations of the curve C are $x = \cos t + t\sin t$, $y = \sin t - t\cos t$, for $0 \le t \le \pi$.

(b) Given $x = \cos t + t\sin t$ .... (1)

Differentiate x with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sin t + \sin t + t\cos t$$
$$= t\cos t$$

For maximum x, $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$

$\therefore\ t\cos t = 0$

$t = 0$ or $\cos t = 0$

$t = \frac{\pi}{2}$

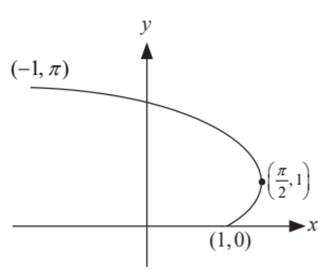
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \cos t - t\sin t$$

When $t = 0$, $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 1 > 0$.

When $t = \frac{\pi}{2}$, $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{\pi}{2} < 0$.

Thus x is maximum when $t = \frac{\pi}{2}$.

(c) The graph of curve C



(d) Given $y = \sin t - t\cos t$ .... (2)

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \cos t - (\cos t - t\sin t)$$
$$= t\sin t$$

Using the Chain Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$
$$= \frac{t\sin t}{t\cos t}$$
$$= \tan t$$

$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x} = \tan t$ .... (3)

Given that Point P on C has parameter p, i.e. $t = p$.
Substitute $t = p$ into (1), (2) and (3).

From (1): $x = \cos p + p\sin p$

From (2): $y = \sin p - p\cos p$

$\therefore$ the coordinates of point $P = (\cos p + p\sin p, \sin p - p\cos p)$

From (3): $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \tan p$

$\therefore$ the gradient of normal at $P = -\dfrac{1}{\tan p}$

Equation of normal at P is

$y - (\sin p - p\cos p) = (-\cot p)(x - (\cos p + p\sin p))$ .... (4)

Given that the normal at P cuts the y-axis at the point $Q(0, k)$, substitute $x = 0$ and $y = k$ into (4)

$$\begin{aligned}
k &= \cot p\cos p + p\cos p + \sin p - p\cos p \\
&= \cot p\cos p + \sin p \\
&= \frac{\cos^2 p}{\sin p} + \sin p \\
&= \frac{\cos^2 p + \sin^2 p}{\sin p} \\
&= \frac{1}{\sin p}
\end{aligned}$$

Given $0 < p < \pi$

$\sin 0 < p < \sin \pi$ $\triangleleft$ introduce sine function all sides

$0 < \sin p \le 1$

$\therefore \dfrac{1}{\sin p} \ge 1$

i.e. $k \ge 1$ (Shown)

Exercise 8

H Exam Style Questions

58
Solution

Given $\dfrac{x^2}{1+x^2}+\dfrac{y^2}{1+y^2}=x^3y^5$

Differentiate both sides with respect to x

$$\frac{2x(1+x^2)-x^2(2x)}{(1+x^2)^2}+\frac{2y\dfrac{\mathrm{d}y}{\mathrm{d}x}(1+y^2)-y^2\left(2y\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)}{(1+y^2)^2}=3x^2y^5+5x^3y^4\frac{\mathrm{d}y}{\mathrm{d}x}$$

Substitute the point $x=1$ and $y=1$ into (2)

$$\frac{4-2}{2^2}+\frac{4\dfrac{\mathrm{d}y}{\mathrm{d}x}-2\dfrac{\mathrm{d}y}{\mathrm{d}x}}{2^2}=3+5\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{1}{2}+\frac{1}{2}\frac{\mathrm{d}y}{\mathrm{d}x}=3+5\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$-\frac{5}{2}=\frac{9}{2}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{5}{9}$$

Gradient of the tangent at $(1, 1)$ is $-\dfrac{5}{9}$

Equation of tangent at $(1,1)$ is

$$y-1=-\frac{5}{9}(x-1)$$

$$9y-9=-5x+5$$

$$5x+9y=14$$

59
Solution

(a) $y = \dfrac{(\ln x)^2}{x}$ .... (1)

Differentiate (1) with respect to x

$$\frac{dy}{dx} = \frac{x \cdot (2\ln x)\left(\frac{1}{x}\right) - (\ln x)^2(1)}{x^2} \quad \lhd \text{ apply quotient rule}$$

$$= \frac{(\ln x)(2 - \ln x)}{x^2}$$

At the turning point, $\dfrac{dy}{dx} = 0$.

$$\therefore \quad \frac{(\ln x)(2-\ln x)}{x^2} = 0$$

$$(\ln x)(2 - \ln x) = 0$$

$$\ln x = 0 \quad \text{or} \quad 2 - \ln x = 0$$

$$x = e^0 \quad \text{or} \quad \ln x = 2$$

$$x = 1 \quad \text{or} \quad x = e^2$$

Substitute $x = 1$ into (1).

$\therefore\ y = \dfrac{(\ln 1)^2}{1} = 0$

The coordinates are $(1, 0)$.

Use First Derivative Test

$\therefore$ the point $(1, 0)$ is a minimum point.

x	1^-	1	1^+
$\frac{dy}{dx}$	$-$	0	$+$
Slope	\	—	/

Substitute $x = e^2$ into (1)

$$\therefore \quad y = \frac{(\ln e^2)^2}{e^2}$$

$$= \frac{(2\ln e)^2}{e^2}$$

$$= \frac{4}{e^2}$$

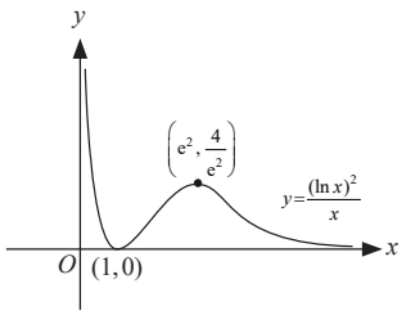
The coordinates are $\left(e^2, \dfrac{4}{e^2}\right)$

Use First Derivative Test

$\therefore$ the point $\left(e^2, \dfrac{4}{e^2}\right)$ is a maximum point.

x	$(e^2)^-$	e^2	$(e^2)^+$
$\frac{dy}{dx}$	$+$	0	$-$
Slope	/	—	\

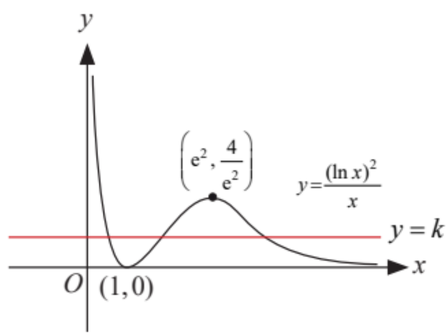
(b)



(c) Given $(\ln x)^2 = kx$

$$\frac{(\ln x)^2}{x} = k$$

The horizontal line $y = k$ will cut the graph of $y = \frac{(\ln x)^2}{x}$ at three distinct points provided $0 < k < \frac{4}{e^2}$.



Hence set of values of k such that the equation $(\ln x)^2 = kx$ has three distinct real roots is $\left\{k \in \mathbb{R} : 0 < k < \frac{4}{e^2}\right\}$.

60
Solution

Given $y = 3x^2 - k^2 \ln\left(\frac{x}{4}\right)$ .... (1)

Differentiate both sides with respect to x

$$\frac{dy}{dx} = 6x - k^2\left(\frac{4}{x}\right)\left(\frac{1}{4}\right)$$

$$= 6x - \frac{k^2}{x}$$

At stationary point, $\frac{dy}{dx} = 0$

i.e. $6x - \frac{k^2}{x} = 0$

$$6x = \frac{k^2}{x}$$

$$x^2 = \frac{k^2}{6}$$

$$x = \pm\frac{k}{\sqrt{6}}$$

Since $k > 0$ and $x > 0$

$$x = \frac{k}{\sqrt{6}}$$

Substitute $x = \frac{k}{\sqrt{6}}$ into (1)

$$y = 3\left(\frac{k}{\sqrt{6}}\right)^2 - k^2 \ln\left(\frac{\frac{k}{\sqrt{6}}}{4}\right)$$

$$= \frac{k^2}{2} - k^2 \ln\left(\frac{k}{4\sqrt{6}}\right)$$

$$= k^2\left(\frac{1}{2} - \ln\frac{\sqrt{6}k}{24}\right)$$

The stationary point is $\left(\frac{\sqrt{6}k}{6}, k^2\left(\frac{1}{2} - \ln\sqrt{6}k + \ln 24\right)\right)$

Use Second Derivative Test

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(6x - \frac{k^2}{x}\right)$$

$$\frac{d^2y}{dx^2} = 6 + \frac{k^2}{x^2}$$

Since $k > 0$ and $x > 0$, $\therefore\ \frac{k^2}{x^2} > 0$

$$\frac{d^2 y}{dx^2} = 6 + \frac{k^2}{x^2} > 0$$

Hence the stationary point $\left(\frac{\sqrt{6}k}{6}, k^2\left(\frac{1}{2} - \ln\sqrt{6}k + \ln 24\right)\right)$ is a minimum point.

61
Solution

$$\mathrm{f}(x)=\frac{2x}{(x+1)(x-2)}$$

$$\mathrm{f}'(x)=\frac{(x+1)(x-2)(2)-(2x)(2x-1)}{\left[(x+1)(x-2)\right]^2}$$

$$=\frac{-2x^2-4}{\left[(x+1)(x-2)\right]^2}$$

$\left[(x+1)(x-2)\right]^2$ is always positive, for all real values of x.

$-2x^2-4$ is always negative, for all real values of x.

$\therefore\ \dfrac{-2x^2-4}{\left[(x+1)(x-2)\right]^2}<0$ is always negative.

Thus, $\mathrm{f}'(x)<0$ for all values of x. (Proved)

62
Solution

(a) Given $x^2 - xy + y^2 - a = 0$ .... (1)

Differentiate (1) with respect to x

$$2x - y - x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$(2y - x)\frac{\mathrm{d}y}{\mathrm{d}x} = y - 2x \quad \text{(Shown) .... (2)}$$

(b) Given that $x = 2\sqrt{3}$ is a tangent to the curve C, which means that the tangent is parallel to y-axis.

i.e. $\frac{\mathrm{d}y}{\mathrm{d}x} \to \infty$

From (2): $(2y - x)\frac{\mathrm{d}y}{\mathrm{d}x} = y - 2x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y - 2x}{2y - x}$$

For $\frac{\mathrm{d}y}{\mathrm{d}x} \to \infty$, let $2y - x = 0$

$2y = x$

When $x = 2\sqrt{3}$, $\quad 2y = 2\sqrt{3}$

$\therefore \quad y = \sqrt{3}$

Substitute $x = 2\sqrt{3}$ and $y = \sqrt{3}$ into (1)

$$(2\sqrt{3})^2 - (2\sqrt{3})(\sqrt{3}) + (\sqrt{3})^2 - a = 0$$

$$4(3) - 2\sqrt{3}(\sqrt{3}) + 3 - a = 0$$

Hence $a = 9$

(c) At stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

From (2): $\quad (2y - x)(0) = y - 2x$

$0 = y - 2x$

$y = 2x$ .... (3)

Substitute (3) into (1)

$$x^2 - 2x^2 + 4x^2 - 9 = 0$$

$$3x^2 - 9 = 0$$

$$x = \pm\sqrt{3}$$

Substitute $x = \sqrt{3}$ into (3): $y = 2\sqrt{3}$

Substitute $x = -\sqrt{3}$ into (3): $y = -2\sqrt{3}$

The exact coordinates of the stationary points are $(\sqrt{3}, 2\sqrt{3})$ and $(-\sqrt{3}, -2\sqrt{3})$

(d) Differentiate (3) with respect to x

$$(2y-x)\frac{\mathrm{d}y}{\mathrm{d}x}=y-2x$$

$$\left(2\frac{\mathrm{d}y}{\mathrm{d}x}-1\right)\frac{\mathrm{d}y}{\mathrm{d}x}+(2y-x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}y}{\mathrm{d}x}-2$$

At stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$.

$$\big(2(0)-1\big)(0)+(2y-x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0-2$$

$$(2y-x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2$$

$$\therefore \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{-2}{(2y-x)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

At $(\sqrt{3}, 2\sqrt{3})$, substitute $x=\sqrt{3}$ and $y=2\sqrt{3}$ into (4)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{-2}{(2y-x)}<0.$$

The turning point $(\sqrt{3}, 2\sqrt{3})$ is a maximum point.

At $(-\sqrt{3},-2\sqrt{3})$, substitute $x=-\sqrt{3}$ and $y=-2\sqrt{3}$ into (4)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{-2}{(2y-x)}>0$$

The turning point $(\sqrt{3}, 2\sqrt{3})$ is a minimum point.

63
Solution

(a) Let $x^2 - 3xy + 2y^2 = 6$ .... (1)

Differentiate (1) with respect to x

$$2x - 3x\frac{\mathrm{d}y}{\mathrm{d}x} - 3y + 4y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$(3x-4y)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 3y$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x-3y}{3x-4y} \text{ .... (2)}$$

Substitute the point $(2,3)$ into (2)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2(2)-3(3)}{3(2)-4(3)}$$

$$= \frac{5}{6}$$

Equation of tangent at (2,3)

$$y - 3 = \frac{5}{6}(x-2)$$

$$6y - 18 = 5x - 10$$

$$6y = 5x + 8$$

The equation of the tangent to the curve at $(2,3)$ is $6y = 5x + 8$

(b) For tangent parallel to the x-axis, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

From (2) $\quad \frac{2x-3y}{3x-4y} = 0$

$$2x - 3y = 0$$

$$y = \frac{2}{3}x$$

Substitute $y = \frac{2}{3}x$ into (1).

$$x^2 - 3x\left(\frac{2}{3}x\right) + 2\left(\frac{2}{3}x\right)^2 = 6$$

$$-9x^2 + 8x^2 = 54$$

$$x^2 = -54$$

Since $x^2 \geq 0, x \in \mathbb{R}$. There is no point on the curve at which the tangent is parallel to the x-axis. (Shown)

(c) Let $y = a$, where a is a constant .... (3)

Substitute (3) into (1) gives

$$x^2 - 3xa + 2a^2 = 6$$

$\therefore\ x^2 - 3xa + (2a^2 - 6) = 0$

Using the Discriminant, $D = (-3a)^2 - 4(2a^2 - 6)$

$= a^2 + 24$

For any real values of a, $a^2 \geq 0$. $\therefore\ a^2 + 24$ is positive.

Since discriminant, $D > 0$, $\therefore\ y = a$ and curve intersect at 2 distinct points. (Shown)

64
Solution

Let $x^2 + y^2 = y(x-3)$ .... (1)

Differentiate (1) with respect to x

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = y + \frac{\mathrm{d}y}{\mathrm{d}x}(x-3)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}(2y - x + 3) = y - 2x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-2x}{2y-x+3}$$

For tangent to be parallel to y-axis, $\frac{\mathrm{d}y}{\mathrm{d}x} \to \infty$.

Let $2y - x + 3 = 0$

$$x = 2y + 3 \text{ .... (2)}$$

Substitute (2) into (1)

$$(2y+3)^2 + y^2 = y(2y)$$

$$3y^2 + 12y + 9 = 0$$

$$(y+3)(y+1) = 0$$

$$y = -3 \text{ or } -1$$

Substitute $y = -3$ into (2): $x = -3$

Substitute $y = -1$ into (2): $x = 1$

The points at which the tangents are parallel to the y-axis are $(-3, -3)$ and $(1, -1)$.

65
Solution

(a) Given $x = a\cos^2\theta$ .... (1)

and $y = a\sin^3\theta$ .... (2)

Differentiate (1) with respect to θ

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -2a\cos\theta\sin\theta$$

Differentiate (2) with respect to θ

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta$$

Using the Chain Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}}$$

$$= \frac{3a\sin^2\theta\cos\theta}{-2\sin\theta\cos\theta}$$

$$= -\frac{3}{2}\sin\theta$$

$\therefore \dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{3}{2}\sin\theta$ .... (3)

When the tangent to C at the point where C crosses the x-axis, i.e. $y = 0$.

Substitute $y = 0$ into (1)

$$a\sin^3\theta = 0$$

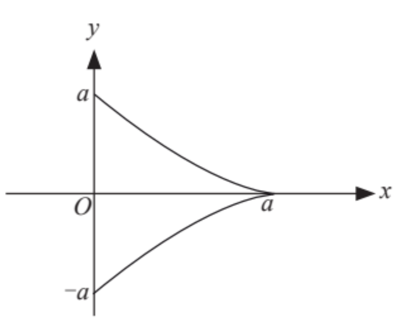
$$\sin\theta = 0$$

Substitute $\sin\theta = 0$ into (3)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

Since $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, i.e. the gradient is 0. The tangent at the x-intercept is parallel to the x-axis.

(b) The graph of curve C



66
Solution

(a) Given $x = 2\cos^3\theta$ .... (1)

and $y = 2\sin^3\theta$ .... (2)

Differentiate (1) with respect to θ

$$\frac{dx}{d\theta} = 3(2)\cos^2\theta(-\sin\theta)$$
$$= -6\sin\theta\cos^2\theta$$

Differentiate (2) with respect to θ

$$\frac{dy}{d\theta} = 3(2)\sin^2\theta\cos\theta$$
$$= 6\sin^2\theta\cos\theta$$

Using the Chain Rule,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$
$$= \frac{6\sin^2\theta\cos\theta}{-6\sin\theta\cos^2\theta}$$
$$= -\frac{\sin\theta}{\cos\theta}$$

$\therefore$ gradient of tangent with parameter $\theta = -\frac{\sin\theta}{\cos\theta}$

Hence, gradient of normal with parameter $\theta = \frac{\cos\theta}{\sin\theta}$

Equation of normal to the curve at $(2\cos^3\theta, 2\sin^3\theta)$ is

$$y - (2\sin^3\theta) = \frac{\cos\theta}{\sin\theta}(x - 2\cos^3\theta)$$
$$y\sin\theta - 2\sin^4\theta = x\cos\theta - 2\cos^4\theta$$
$$y\sin\theta = x\cos\theta + 2(\sin^4\theta - \cos^4\theta) \quad \text{(Shown) .... (3)}$$

(b) At point Q, $\theta = \frac{\pi}{6}$.

Substitute $\theta = \frac{\pi}{6}$ into (3)

$$y\sin\frac{\pi}{6} = x\cos\frac{\pi}{6} + 2\left(\sin^4\frac{\pi}{6} - \cos^4\frac{\pi}{6}\right)$$
$$y\left(\frac{1}{2}\right) = x\left(\frac{\sqrt{3}}{2}\right) + 2\left(\left(\frac{1}{2}\right)^4 - \left(\frac{\sqrt{3}}{2}\right)^4\right)$$
$$y = \sqrt{3}x - 2$$

$\therefore$ equation of normal to the curve at Q, i.e. $\theta = \frac{\pi}{6}$ is $y = \sqrt{3}x - 2$ .... (4)

When the normal to the curve at Q cuts C again at P, i.e. $\theta = p$.

Substitute (1) and (2) into (4)

$$2\sin^3\theta = \sqrt{3}(2\cos^3\theta) - 2$$

When $\theta = p$,

$$2\sin^3 p = \sqrt{3}(2\cos^3 p) - 2$$

$$\sin^3 p - \sqrt{3}\cos^3 p + 1 = 0 \qquad \text{(Shown)}$$

Use GC to solve $\sin^3 p - \sqrt{3}\cos^3 p + 1 = 0$.

$p = -0.7445633$ or 0.52359878 (rejected, since when $p = 0.52359878$, it gives point Q)

Substitute $p = -0.7445633$ into (1) and (2).

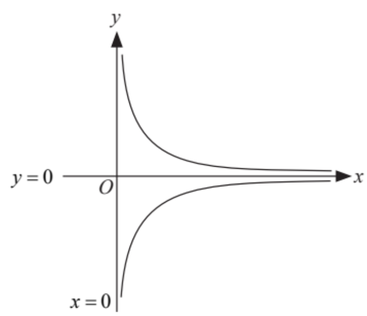
From (1): $x = 2\cos^3(-0.7445633) = 0.795$

From (2): $y = 2\sin^3(-0.7445633) = -0.622$

$\therefore$ the coordinates of P are $(0.795, -0.622)$

67
Solution

(a) The graph of curve C



(b) Given $x = at^2$ .... (1)

and $y = \dfrac{a}{t}$ .... (2)

Differentiate (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2at$$

Differentiate (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = -\frac{a}{t^2}$$

Using the Chain Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

$$= -\frac{a}{t^2} \times \frac{1}{2at}$$

$$= -\frac{1}{2t^3} \quad \text{(Shown)} \text{ .... (3)}$$

(c) At the point $P\left(ap^2, \dfrac{a}{p}\right)$, where $t = p$.

Substitute $t = p$ into (3)

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2p^3}$$

Equation of tangent at P is given by

$$y - \frac{a}{p} = -\frac{1}{2p^3}(x - ap^2)$$

$$2p^3 y - 2ap^2 = -x + ap^2$$

$$2p^3 y + x = 3ap^2 \quad \text{(Shown)} \text{ .... (4)}$$

(d) Given that the tangent to the curve at P cuts the curve again at $Q\left(aq^2, \ \frac{a}{q}\right)$,

substitute $x = aq^2$ and $y = \frac{a}{q}$ into (4).

i.e. $\quad 2p^3\frac{a}{q} + aq^2 = 3ap^2$

$$q^3 - 3p^2q + 2p^3 = 0$$

$$(q-p)^2(q+2p) = 0$$

$$q = p \quad \text{or} \quad q = -2p$$

Since $q \neq p$, thus $q = -2p$.

68
Solution

(a) Given $x = 1 - \theta\cos\theta$ .... (1)

and $y = 1 - \cos\theta$ .... (2)

Differentiate (1) with respect to θ

$$\frac{dx}{d\theta} = 0 - (-\theta\sin\theta + \cos\theta)$$
$$= \theta\sin\theta - \cos\theta$$

Differentiate (2) with respect to θ

$$\frac{dy}{d\theta} = \sin\theta$$

Using the Chain Rule,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$
$$= \frac{\sin\theta}{\theta\sin\theta - \cos\theta}$$

$\therefore \ \dfrac{dy}{dx} = \dfrac{\sin\theta}{\theta\sin\theta - \cos\theta}$ .... (3)

As $\theta \to 0, \ \sin\theta \to 0. \ \therefore \ \dfrac{dy}{dx} \to 0.$

As $\theta \to \pi, , \ \sin\theta \to 0. \ \therefore \ \dfrac{dy}{dx} \to 0.$

The gradient of the tangents tends to zero.
The tangents tends to become horizontal lines.

(b) When $\theta = 0,$ substitute $\theta = 0$ into (1) and (2)

$x = 1 - (0)\cos(0) = 1$

$y = 1 - \cos(0) = 0$

The point is (1, 0)

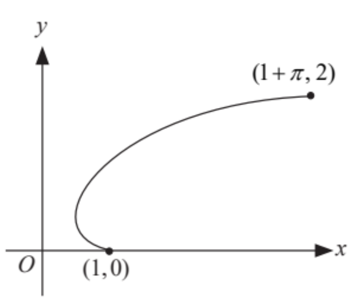
When $t = \pi,$ substitute $\theta = \pi$ into (1) and (2)

$x = 1 - \pi\cos\pi = 1 + \pi$

$y = 1 - (-1) = 2$

The point is $(1 + \pi, 2)$

The graph of C



(c) When $\theta = \frac{\pi}{2}$, substitute $\theta = \frac{\pi}{2}$ into (1) and (2)

From (1): $x = 1 - \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) = 1$

From (2): $y = 1 - \cos\left(\frac{\pi}{2}\right) = 1$

The point at $\theta = \frac{\pi}{2}$ is (1, 1).

Substitute $\theta = \frac{\pi}{2}$ into (3)

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=\frac{\pi}{2}} = \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right)}$$
$$= \frac{2}{\pi}$$

The gradient of normal at $\theta = \frac{\pi}{2}$ is $-\frac{\pi}{2}$

Equation of normal at $\theta = \frac{\pi}{2}$ is

$$y - 1 = -\frac{\pi}{2}(x - 1)$$
$$y = -\frac{\pi}{2}x + \left(1 + \frac{\pi}{2}\right)$$

For point Q, let $x = 0$, substitute $x = 0$ into (4)

$\therefore\ y = 1 + \frac{\pi}{2}$

Coordinates of $Q = \left(0,\ 1 + \frac{\pi}{2}\right)$

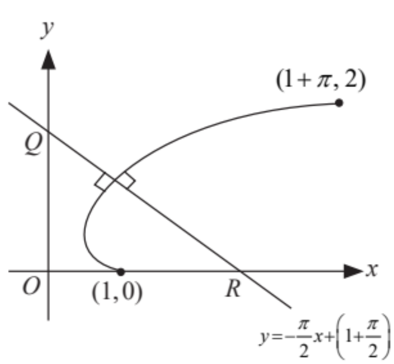
For point R, let $y = 0$, substitute $y = 0$ into (4)

$$x = \frac{2}{\pi}\left(1 + \frac{\pi}{2}\right)$$
$$= \frac{2}{\pi} + 1$$

Coordinates of $R = \left(\frac{2}{\pi} + 1, 0\right)$

Area of ΔQOR

$$= \frac{1}{2}(OR)(OQ)$$
$$= \frac{1}{2}\left(\frac{2}{\pi} + 1\right)\left(1 + \frac{\pi}{2}\right)$$
$$= \frac{(\pi + 2)^2}{4\pi}$$



69
Solution

(a) Given $x = t + \frac{1}{t}$ .... (1)

and $y = t - \frac{1}{t}$ .... (2)

At $t = p$, substitute $t = p$ into (1) and (2)

$x = p + \frac{1}{p}$ and $y = p - \frac{1}{p}$

The coordinates of P are $\left(p + \frac{1}{p},\ p - \frac{1}{p}\right)$

At $t = q$, substitute $t = q$ into (1) and (2)

$x = q + \frac{1}{q}$ and $y = q - \frac{1}{q}$

The coordinates of Q are $\left(q + \frac{1}{q},\ q - \frac{1}{q}\right)$

Gradient PQ

$$= \frac{\left(p - \frac{1}{p}\right) - \left(q - \frac{1}{q}\right)}{\left(p + \frac{1}{p}\right) - \left(q + \frac{1}{q}\right)}$$

$$= \frac{(p-q) - \left(\frac{1}{p} - \frac{1}{q}\right)}{(p-q) + \left(\frac{1}{p} - \frac{1}{q}\right)}$$

$$= \frac{(p-q) - \left(\frac{q-p}{pq}\right)}{(p-q) + \left(\frac{q-p}{pq}\right)} \qquad \triangleleft \text{ divide each term by } (p-q)$$

$$= \frac{1 + \left(\frac{1}{pq}\right)}{1 - \left(\frac{1}{pq}\right)}$$

$$= \frac{pq+1}{pq-1} \quad \text{(Shown)}$$

(b) Differentiate (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - \frac{1}{t^2}$$

$$= \frac{t^2 - 1}{t^2}$$

Differentiate (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 + \frac{1}{t^2}$$

$$= \frac{t^2 + 1}{t^2}$$

Using the Chain Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

$$= \frac{t^2 + 1}{t^2} \times \frac{t^2}{t^2 - 1}$$

$$= \frac{t^2 + 1}{t^2 - 1}$$

At $t = p$, $\qquad \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{p^2 + 1}{p^2 - 1}$

Thus gradient of normal at P is $\dfrac{1 - p^2}{1 + p^2}$ (Shown) .... (3)

(c) At $t = r$, substitute $t = r$ into (1) and (2)

$x = r + \dfrac{1}{r}$ and $y = r - \dfrac{1}{r}$

The coordinates of R are $\left(r + \dfrac{1}{r},\ r - \dfrac{1}{r} \right)$

Gradient PR

$= \dfrac{pr + 1}{pr - 1}$ $\qquad \triangleleft$ replace $q = r$ in $\dfrac{pq + 1}{pq - 1}$ from **(a)** .... (4)

Note that PR is parallel to the normal at P. Thus gradient PR = gradient of normal at P

i.e. (3) = (4)

$$\therefore \qquad \frac{pr + 1}{pr - 1} = \frac{1 - p^2}{1 + p^2}$$

$$(pr + 1)(1 + p^2) = (2 - p^2)(pr - 1)$$

$$pr + p^3 r + 1 + p^2 = pr - p^3 r - 1 + p^2$$

$$p^3 r = -1$$

$$r = -\frac{1}{p^3} \quad \text{(Shown)}$$

(d) At $t=-p$, substitute $t=-p$ into (1) and (2)

$x=-p+\frac{1}{-p}$ and $y=-p-\frac{1}{-p}$

The coordinates of S are $\left(-p-\frac{1}{p},\ -p+\frac{1}{p}\right)$

Gradient PS

$=\frac{p(-p)+1}{p(-p)-1}$ ◁ replace $q=-p$ in $\frac{pq+1}{pq-1}$ from **(a)**

$=-\frac{1-p^2}{1+p^2}$

Gradient SR

$=\frac{pr+1}{pr-1}$ ◁ replace $q=r$ in $\frac{pq+1}{pq-1}$ from **(a)**

$=\frac{p(-p^{-3})+1}{p(-p^{-3})-1}$ ◁ replace $r=-\frac{1}{p^3}$ from **(c)**

$=\frac{p^{-2}+1}{p^{-2}-1}$

$=\frac{p^{-2}+1}{p^{-2}-1}\times\frac{p^2}{p^2}$

$=\frac{1+p^2}{1-p^2}$

Gradient $PS\times$Gradient SR

$=-\frac{1-p^2}{1+p^2}\times\frac{1+p^2}{1-p^2}$

$=-1$

PS is perpendicular to RS. (Shown)

70
Solution

(a) Given $x = e^t \sin t$ .... (1)

and $y = e^{-t} \cos t$ .... (2)

Differentiate (1) with respect to t

$$\frac{dx}{dt} = e^t \sin t + e^t \cos t$$
$$= e^t (\sin t + \cos t)$$

Differentiate (2) with respect to t

$$\frac{dy}{dt} = e^{-t}(-\sin t) - e^{-t} \cos t$$
$$= -e^{-t}(\sin t + \cos t)$$

Using the Chain Rule,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$
$$= \frac{-e^{-t}(\sin t + \cos t)}{e^t(\sin t + \cos t)}$$
$$= -\frac{1}{e^{2t}}$$

$\therefore \frac{dy}{dx} = -\frac{1}{e^{2t}}$

For $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, $\frac{dy}{dx} = -\frac{1}{e^{2t}} < 0$

Since $\frac{dy}{dx} \ne 0$, therefore C has no stationary point.

(b) Equation of tangent at parameter t is

$$y - e^{-t} \cos t = -\frac{1}{e^{2t}}(x - e^t \sin t) \text{ .... (3)}$$

Given that the tangent to pass through origin, substitute $x = 0$ and $y = 0$ into (3)

$$0 - e^{-t} \cos t = -\frac{1}{e^{2t}}(0 - e^{-t} \sin t)$$
$$-e^{-t} \cos t = e^{-t} \sin t$$
$$\tan t = -1$$
$$t = -\frac{\pi}{4} \quad \text{where } -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

Substitute $t=-\frac{\pi}{4}$ into (3)

$$y-\mathrm{e}^{\frac{\pi}{4}}\cos\left(-\frac{\pi}{4}\right)=-\frac{1}{2\left(-\frac{\pi}{4}\right)}\left(x-\mathrm{e}^{-\frac{\pi}{4}}\sin\left(-\frac{\pi}{4}\right)\right)$$

$$y-\frac{1}{\sqrt{2}}\mathrm{e}^{\frac{\pi}{4}}=-\mathrm{e}^{\frac{\pi}{2}}\left(x+\frac{1}{\sqrt{2}}\mathrm{e}^{-\frac{\pi}{4}}\right)$$

$$y=-\mathrm{e}^{\frac{\pi}{2}}x$$

The equation of tangent is $y=-\mathrm{e}^{\frac{\pi}{2}}x$

71
Solution

(a) Given $x = \sin t \cos t$ .... (1)

and $y = \cos\left(t + \frac{\pi}{4}\right)$ .... (2)

Differentiate (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \cos^2 t - \sin^2 t$$

Differentiate (2) with respect to t

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\cos t \cos\frac{\pi}{4} - \sin t \sin\frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\cos t - \sin t) \\ &= -\frac{\sqrt{2}}{2}(\sin t + \cos t)\end{aligned}$$

Using the Chain Rule,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{-\frac{\sqrt{2}}{2}(\sin t + \cos t)}{\cos^2 t - \sin^2 t} \\ &= -\frac{\sin t + \cos t}{\sqrt{2}(\cos t + \sin t)(\cos t - \sin t)} \\ &= \frac{1}{\sqrt{2}(\sin t - \cos t)}\end{aligned}$$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{2}(\sin t - \cos t)}$ (Shown)

(b) For tangent parallel to y-axis, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined.

Let the denominator of $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{2}(\sin t - \cos t)}$ be zero.

i.e. $\sin t - \cos t = 0$

$$\tan t = 1$$

$$t = \frac{\pi}{4}$$

Substitute $t = \frac{\pi}{4}$ into (1): $x = \sin\frac{\pi}{4}\cos\frac{\pi}{4}$

$$= \frac{1}{2}$$

Equation of tangent is $x = \frac{1}{2}$

(c) Given that the curve cuts the y-axis at the points P and Q, i.e. the x-coordinates of P and Q are zero.

Let $x = 0$.

$\therefore \ \sin t \cos t = 0$

$$\sin t = 0 \quad \text{or} \quad \cos t = 0$$

$$t = 0 \qquad\qquad t = \frac{\pi}{2}$$

Substitute $t = 0$ into (2)

$$y = \cos\left(0 + \frac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{2}$$

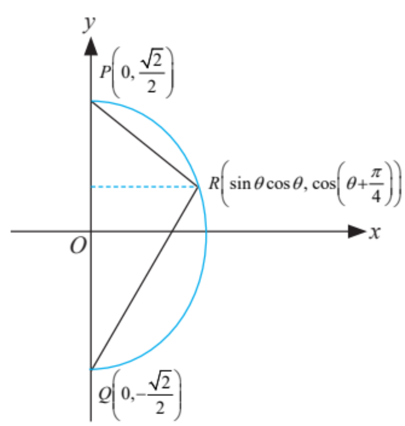
Substitute $t = \frac{\pi}{2}$ into (2)

$$y = \cos\left(\frac{\pi}{2} + \frac{\pi}{4}\right)$$

$$= -\frac{\sqrt{2}}{2}$$

$\therefore$ the coordinates of P and Q are $P\left(0, \frac{\sqrt{2}}{2}\right)$ and $Q\left(0, -\frac{\sqrt{2}}{2}\right)$

(d) Refer to the diagram.



$$\text{Area of triangle } PQR = \frac{1}{2} \times 2\left(\frac{\sqrt{2}}{2}\right) \times \sin\theta\cos\theta$$

$$= \frac{\sqrt{2}}{2}\left(\frac{1}{2}\sin 2\theta\right)$$

$$= \frac{\sin 2\theta}{2\sqrt{2}}$$

$$= \frac{\sqrt{2}\sin 2\theta}{4} \quad \text{(Shown)}$$

Method 1 (Use calculus)

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sqrt{2}\sin 2\theta}{4}\right) = \frac{2\sqrt{2}\cos 2\theta}{4}$$

At stationary, $\dfrac{2\sqrt{2}\cos 2\theta}{4} = 0$

$$\cos 2\theta = 0$$

$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

Method 2

Since $0 \le \sin 2\theta \le 1$ for $0 \le \theta \le \dfrac{\pi}{2}$,

Max area of triangle PQR occurs when $\sin 2\theta = 1$.

$$\therefore \quad \sin 2\theta = 1$$

$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

The value of θ is $\dfrac{\pi}{4}$.

72
Solution

(a) Given $x = 5a\sec\theta$ .... (1)

and $y = 3a\tan\theta$ .... (2)

From (2), let $y = 0$.

$3a\tan\theta = 0$

$\theta = 0$

Substitute $\theta = 0$ into (1)

$\therefore\ x = 5a$

Coordinates of x-intercept of C is $(5a, 0)$.

(b) Differentiate (1) with respect to θ

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 5a\sec\theta\tan\theta$$

Differentiate (2) with respect to θ

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sec^2\theta$$

Using the Chain Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

$$= \frac{3a\sec^2\theta}{5a\sec\theta\tan\theta}$$

$$= \frac{3}{5}\left(\frac{1}{\cos\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$= \frac{3}{5}\operatorname{cosec}\theta$$

$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{5}\operatorname{cosec}\theta$ (Shown) .... (3)

(c) Gradient of normal $= \dfrac{-1}{\frac{3}{5}\operatorname{cosec}\theta}$

$= -\frac{5}{3}\sin\theta$ .... (4)

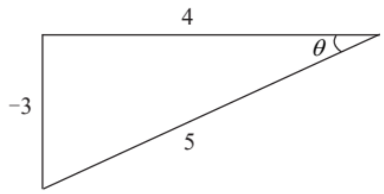
Given that normal makes an angle $\frac{\pi}{4}$ with the positive x-axis,

i.e. gradient of normal $= \tan\frac{\pi}{4}$

$\therefore\ -\frac{5}{3}\sin\theta = 1$

$\sin\theta = -\frac{5}{3}$

From the diagram,



$$\cos\theta = \frac{4}{5}$$

$$\tan\theta = -\frac{3}{4}$$

From (1): $x = 5a\sec\theta$

$$= 5a\left(\frac{1}{\cos\theta}\right) \quad \triangleleft \text{ substitute } \cos\theta = \frac{4}{5}$$

$$= 5a\left(\frac{5}{4}\right)$$

$$= \frac{25}{4}a \text{ and}$$

From (2): $y = 3a\tan\theta$

$$y = 3a\left(-\frac{3}{4}\right) \quad \triangleleft \text{ substitute } \tan\theta = -\frac{3}{4}$$

$$= -\frac{9}{4}a$$

The coordinates of the point is $\left(\frac{25}{4}a, -\frac{9}{4}a\right)$

(d) At $\theta = \frac{\pi}{4}$, substitute $\theta = \frac{\pi}{4}$ into (1), (2) and (3)

From (1): $x = 5a\sec\left(\frac{\pi}{4}\right) = 5\sqrt{2}a$

From (2): $y = 3a\tan\left(\frac{\pi}{4}\right) = 3a$

From (3): $\frac{dy}{dx} = \frac{3}{5}\operatorname{cosec}\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{5}$

At $\theta = \frac{\pi}{4}$ the coordinates are $(5\sqrt{2}a, 3a)$ and the gradient of the tangent is $\frac{3\sqrt{2}}{5}$

Equation of tangent at $\theta = \frac{\pi}{4}$ is

$$y - 3a = \frac{3\sqrt{2}}{5}(x - 5\sqrt{2}a) \text{ .... } (5)$$

The tangent meets the y-axis at S, i.e. when $x = 0$.
Substitute $x = 0$ into (5)

$$y - 3a = \frac{3\sqrt{2}}{5}(0 - 5\sqrt{2}a)$$

$$y = \frac{3\sqrt{2}}{5}(-5\sqrt{2}a) + 3a$$

$$= -3a$$

$\therefore\ T(0, -3a)$

Gradient of the normal at $\theta = \frac{\pi}{4}$ is $\frac{-5}{3\sqrt{2}}$

Equation of normal at $\theta = \frac{\pi}{4}$ is

$$y - 3a = \frac{-5}{3\sqrt{2}}\left(x - 5\sqrt{2}a\right)$$

The normal meets the y-axis at T, i.e. when $x = 0$.

$$y - 3a = \frac{-5}{3\sqrt{2}}\left(0 - 5\sqrt{2}a\right)$$

$$y = \frac{-5}{3\sqrt{2}}(5\sqrt{2}a) + 3a$$

$$= \frac{34}{3}a$$

$\therefore\ T\left(0, \frac{34}{3}a\right)$

Distance ST

$= \frac{34}{3}a + 3a$

$= \frac{43}{3}a$, where $k = \frac{43}{3}$

73
Solution

(a) At stationary $f'(x) = 0$, i.e. the x-intercept of the graph of $y = f'(x)$.
From the graph, $x = -a$ and $x = a$

x	$(-a)^-$	$-a$	$(-a)^+$
$f'(x)$	$+$	0	$-$

From the table, we note that $f'(-a) < 0$.

$\therefore$ when $x = -a$, the point is a maximum point.

x	a^-	a	a^+
$f'(x)$	$-$	0	$+$

From the table, we note that $f'(-a) > 0$.

$\therefore$ when $x = -a$, the point is a minimum point.

(b)(i) For $y = f(x)$ is strictly increasing, it occurs when $f'(x) > 0$.
From the graph, this occurs when $x < -a$ or $x > a$.

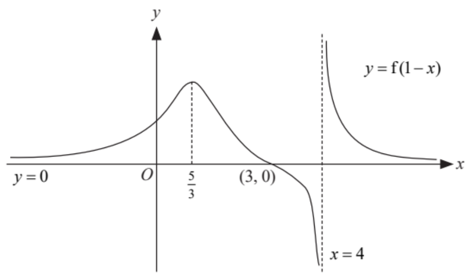
$\therefore \; x < -a$ or $x > a$

(ii) For $y = f(x)$ concave downwards, it occurs when $f''(x) < 0$.
Refer to the graph given. This occurs when graph of $y = f'(x)$ is downward slope.

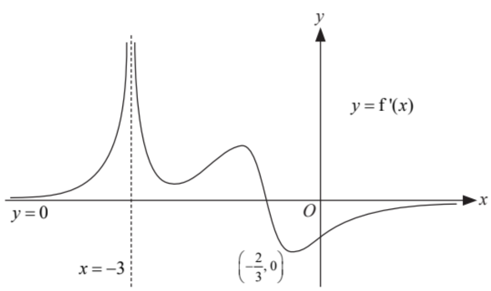
$\therefore \; x < 0$

74
Solution

(a) The graph of $y = f(1-x)$

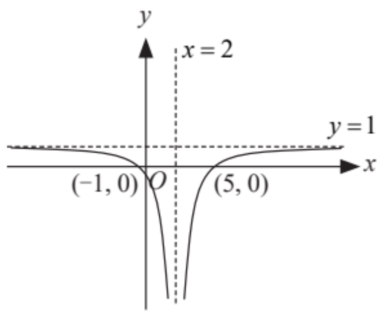


(b) The graph of $y = f'(x)$



75
Solution

(a) The graph of $y = f'(x)$

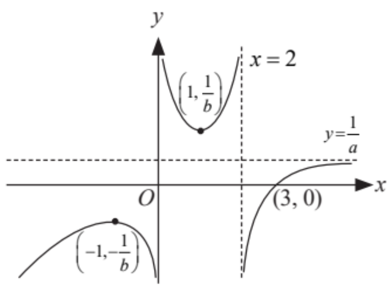


(b)(i) The range of values of x for which $y = f(x)$ is strictly decreasing is $-1 < x < 2$ or $2 < x < 5$.

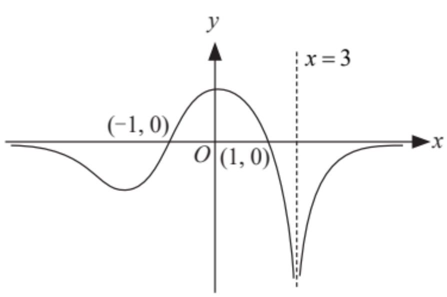
(ii) The range of values of x for which $y = f(x)$ is concave upwards is $x > 2$.

76
Solution

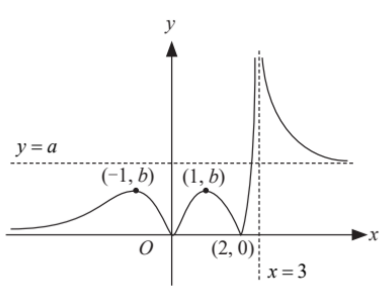
(a) The graph of $y = \dfrac{1}{f(x)}$



(b) The graph of $y = f'(x)$

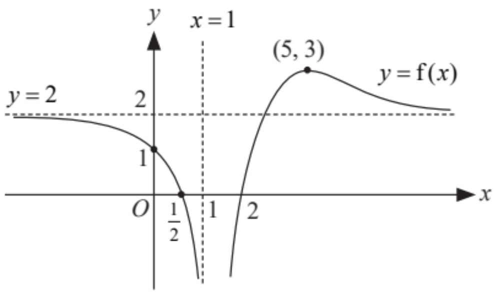


(c) The graph of $y = |f(x)|$



77

Solution



78
Solution

(a) Given $x = 2t - \dfrac{1}{t}$ .... (1)

and $y = t + \dfrac{1}{t}$ .... (2)

Differentiate (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2 + \frac{1}{t^2}$$

Differentiate (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - \frac{1}{t^2}$$

Using the Chain Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

$$= \frac{1 - \frac{1}{t^2}}{2 + \frac{1}{t^2}}$$

$$= \frac{t^2 - 1}{2t^2 + 1}$$

$$= \frac{\frac{1}{2}(2t^2 + 1) - \frac{3}{2}}{2t^2 + 1}$$

$$= \frac{(2t^2 + 1)}{2(2t^2 + 1)} - \frac{3}{2(2t^2 + 1)}$$

$$= \frac{1}{2} - \frac{3}{2(2t^2 + 1)} \qquad \text{(Shown)}$$

(b) Consider $\quad 2t^2 > 0,\quad$ for all $t,\ t \neq 0$

$$2t^2 + 1 > 1 \quad \triangleleft \text{ add 1 on both sides}$$

$$0 < \frac{1}{2t^2 + 1} < 1 \quad \triangleleft \text{ multiplicative inverse}$$

$$0 < \frac{1}{2(2t^2 + 1)} < \frac{1}{2} \quad \triangleleft \text{ multiplty } \frac{1}{2} \text{ on both sides}$$

$$-\frac{1}{2} < -\frac{1}{2(2t^2 + 1)} < 0$$

$$-\frac{3}{2} < -\frac{3}{2(2t^2 + 1)} < 0$$

$$-1 < \frac{1}{2} - \frac{3}{2(2t^2 + 1)} < \frac{1}{2}$$

$$-1 < \frac{\mathrm{d}y}{\mathrm{d}x} < \frac{1}{2}$$

$\therefore$ the range is $-1 < \dfrac{\mathrm{d}y}{\mathrm{d}x} < \dfrac{1}{2}$.

79
Solution

Given $x = \ln(2t)$ .... (1)

and $y = \tan^{-1}(2t)$ .... (2)

(a) As $x \to \infty, t \to \infty$, so $y \to \dfrac{\pi}{2}$,

i.e. curve C approaches the line $y = \dfrac{\pi}{2}$.

As $x \to -\infty, t \to 0$, so $y \to 0$,

i.e. curve C approaches the line $y = 0$

The curve C has horizontal asymptotes $y = 0$ and $y = \dfrac{\pi}{2}$.

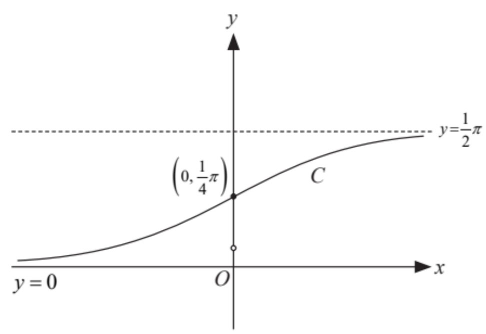
Substitute $x = 0$ into (1)

$$\ln(2t) = 0$$
$$2t = 1$$

$\therefore\ y = \tan^{-1}(1) = \dfrac{\pi}{4}$

Curve C has y-intercept $\left(0, \dfrac{\pi}{4}\right)$.

Since $t > 0$, then $y \neq 0$, so the curve C has no x-intercept.



(b) Differentiate (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{2t} = \frac{1}{t}$$

Differentiate (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{2}{1+(2t)^2} = \frac{2}{1+4t^2}$$

Using the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$= \frac{2t}{1+4t^2}$$

$$= \frac{2t}{1+(2t)^2} \quad \text{.........................} \ (3)$$

When $y = p$, substitute $y = p$ into (2)

$$\tan^{-1}(2t) = p$$

$$2t = \tan p \quad \text{.......................} \ (4)$$

Substitute (4) into (3)

$$\frac{dy}{dx} = \frac{\tan p}{1+\tan^2 p}$$

$$= \frac{\tan p}{\sec^2 p}$$

$$= \left(\frac{\sin p}{\cos p}\right)(\cos^2 p)$$

$$= \sin p \cos p$$

$$= \frac{2\sin p \cos p}{2}$$

$$= \frac{1}{2}\sin 2p$$

$\therefore$ Gradient of C at $y = p$ is $\frac{1}{2}\sin 2p$ (Shown), where $k = 2$

(c) When $p = \frac{\pi}{4}$, i.e. $y = \frac{\pi}{4}$.

Substitute $y = \frac{\pi}{4}$ into (2)

$$\frac{\pi}{4} = \tan^{-1}(2t)$$

$$\therefore \quad \tan\frac{\pi}{4} = 2t$$

$$1 = 2t$$

$$t = \frac{1}{2}$$

Substitute $t = \frac{1}{2}$ into (1) and (3)

$$\therefore \quad x = \ln 1 = 0$$

$$\frac{dy}{dx} = \frac{1}{1+1^2} = \frac{1}{2}$$

Equation of tangent at $\left(0, \frac{\pi}{4}\right)$ is

$$y - \frac{\pi}{4} = \frac{1}{2}(x - 0)$$

$$y = \frac{1}{2}x + \frac{\pi}{4} \text{ .... (1)}$$

Equation of normal at $\left(0, \frac{\pi}{4}\right)$ is

$$y - \frac{\pi}{4} = -2(x - 0)$$

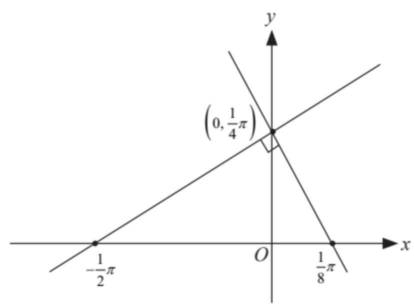
$$y = -2x + \frac{\pi}{4} \text{ .... (2)}$$

Substitute $y = 0$ into (1) and (2)

From (1): $0 = \frac{1}{2}x + \frac{\pi}{4}$

$$x = -\frac{\pi}{2}$$

The coordinates are $\left(-\frac{\pi}{2}, 0\right)$.



From (2): $0 = -2x + \frac{\pi}{4}$

$$x = \frac{\pi}{8}$$

The coordinates are $\left(-\frac{\pi}{8}, 0\right)$.

Area of triangle bounded by the x-axis, the tangent and normal to C

$$= \frac{1}{2}\left(\frac{\pi}{4}\right)\left[\frac{\pi}{8} - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{5\pi^2}{64} \text{ units}^2$$